Second gradient theory

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Duality in mechanics (point masses)

Dynamics of a point mass is driven by

- The balance of forces: the external mechanical actions on the mass can be described by a vector F suc that

$$m\gamma = F$$

or by

- the principle of virtual powers: the external mechanical actions on the mass can be described by a linear form $\mathcal{P}$ such that

$$\forall \widetilde{V} \in \mathbb{R}^3,\ m\gamma \cdot \widetilde{V} = \mathcal{P}(\widetilde{V})$$

- As well known any linear form on $\mathbb{R}^3$ can be identify to a scalar product : $\mathcal{P}(\widetilde{V})$ has the form $\mathcal{P}(\widetilde{V}) = F \cdot \widetilde{V}$ and the two principles are equivalent.

- Generalization to finite number of particles is straightforward.

Duality in mechanics (rigid solids)

- The displacement of a rigid solid is an isometry.
- The only possible velocity fields $\widetilde{V}$ have to satisfy the equiprojectivity property

$$\forall (x,y) \in (\mathbb{R}^3)^2,\ (\widetilde{V}(x) - \widetilde{V}(y)) \cdot (x - y) = 0$$

- This makes a dimension 6 vector space. Indeed

$$(\Omega, W) \mapsto (V : x \mapsto W + \Omega \cdot x)$$

  is an isomorphism with the set $SKEW \times \mathbb{R}^3$. Its dual has a similar structure

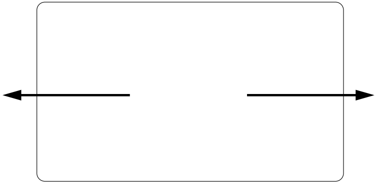
$$\mathcal{P}(\widetilde{V}) = M \cdot \Omega + R \cdot W$$

- (M, R) is a torque-resultant representation of mechanical actions
- Generalization to finite number of rigid solids is straightforward.

  Let us show that the concept of forces is here both unsufficient and superfluous:

Superfluous

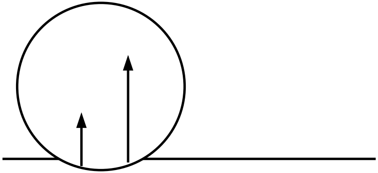
Two opposite forces



- have no physical meaning *inside the theory*
- no power is expanded in any possible motion
- is 0 in the dual of the space of rigid motions

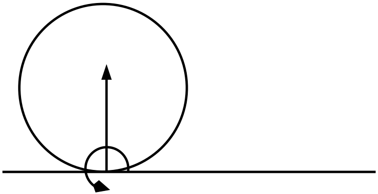
Unsufficient

A wheel on sand



Unsufficient

A wheel on sand in rigid mechanics



- The applied torque at the contact point is not a force.
- It corresponds to some expanded power
- It is a non trivial element of the dual of rigid motions

Duality

Conclusion

- The PVP is equivalent to the momentum balance in simple situations
- It is more precise for systems with “sophisticated” kinematics
- In continuum mechanics : velocity fields belong to a space of smooth functions. Elements of the dual are *distributions*.

Second gradient theory

There are two way for constructing the theory:

1. postulating a form for the internal virtual power and deducing boundary actions
2. postulating a form for boundary interactions and stating a representation theorem for internal stresses

Let us start by the first (and easier) method.

Second gradient theory

We assume the following form for internal virtual power

$$\widetilde{\mathcal{P}}^{int}(V) = -\int_{\mathcal{D}} \sum_{i} a_i V_i + \sum_{i,j} b_{ij} \partial_j V_i + \sum_{i,j,k} c_{ijk} \partial_j \partial_k V_i$$

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and apply the principle of virtual power

$$\forall V, \quad \int_{\mathcal{D}} \rho\gamma_i V_i = \widetilde{\mathcal{P}}^{int}(V) + \widetilde{\mathcal{P}}^{ext}(V)$$

to explicit external actions

Second gradient theory

Let us integrate by parts the last term in

$$\widetilde{\mathcal{P}}^{ext}(V) = \int_{\mathcal{D}} \rho\gamma_i V_i - \widetilde{\mathcal{P}}^{int}(V) = \int_{\mathcal{D}} \rho\gamma_i V_i + \int_{\mathcal{D}} b_{ij} V_{i,j} + c_{ijk} V_{i,jk}$$

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$$\widetilde{\mathcal{P}}^{ext}(V) = \int_{\mathcal{D}} \rho \sum_i \gamma_i V_i + \int_{\mathcal{D}} b_{ij} V_{i,j} - c_{ijk,k} V_{i,j} + \int_{\partial\mathcal{D}} c_{ijk} n_k V_{i,j}$$

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Setting $\sigma_{ij} = b_{ij} - c_{ijk,k}$,

Second gradient theory

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Let us integrate by parts again

$$\widetilde{\mathcal{P}}^{ext}(V) = \int_{\mathcal{D}} (\rho\gamma_i - \sigma_{ij,j}) V_i + \int_{\partial\mathcal{D}} \sigma_{ij} n_j V_i + c_{ijk} n_k V_{i,j}$$

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Setting

$$f^{ext} = \rho\gamma - div(\sigma),$$

we get

$$\widetilde{\mathcal{P}}^{ext}(V) = \int_{\mathcal{D}} f_i^{ext} V_i + \int_{\partial\mathcal{D}} \sigma_{ij} n_j V_i + c_{ijk} n_k V_{i,j}$$

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Now, let us integrate by parts the last term on the boundary. We need to separate normal and tangent derivatives:

$$V_{i,j} = V_{i,j}^n + V_{i,j}^t, \quad \text{where } V_{i,j}^n = V_{i,\ell} n_\ell n_j, \quad V_{i,j}^t = V_{i,\ell} P_{\ell j}, \quad P_{\ell j} = \delta_{\ell j} - n_\ell n_j$$

Second gradient theory

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Then

$$\int_{\partial\mathcal{D}} c_{ijk} n_k V_{i,j}^t = \int_{\partial\mathcal{D}} c_{ijk} n_k V_{i,\ell} P_{\ell q} P_{qj} = -\int_{\partial\mathcal{D}} (c_{ijk} n_k P_{qj})_{,\ell} P_{\ell q} V_i + \int_{\partial\partial\mathcal{D}} c_{ijk} n_k \nu_j V_i$$

Second gradient theory

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Setting

$$F_i^{ext} = \sigma_{ij} n_j - (c_{ijk} n_k P_{qj})_{,\ell} P_{\ell q},$$
$$\mathcal{F}_i^{ext} = c_{ijk} n_k \nu_j, \qquad \mathcal{G}_i^{ext} = c_{ijk} n_k n_j,$$

we get

$$\widetilde{\mathcal{P}}^{ext}(V) = \int_{\mathcal{D}} f_i^{ext} V_i + \int_{\partial\mathcal{D}} F_i^{ext} V_i + \int_{\partial\mathcal{D}} \mathcal{G}_i^{ext} V_{i,j} n_j + \int_{\partial\partial\mathcal{D}} \mathcal{F}_i^{ext} V_i$$

Second gradient theory

We have obtained

- The balance of momentum in the volume

$$f^{ext} = \rho\gamma - div(\sigma),$$

  The density f^{ext} is a volume density of forces. Here σ plays the role of the Cauchy stress tensor.

- Surface contact forces explicitely depending on the curvature of the boundary

$$F^{ext} = \sigma \cdot n - div^{s}(c \cdot n)$$

  Here σ does not represent surface contact forces. In that sense it is not the Cauchy stress tensor.

- Contact edge forces are present

$$\mathcal{F}^{ext} = (c \cdot n) \cdot \nu$$

  They can play an important role in the global balance of forces.

- A contact action which is not a force is also present.

$$\mathcal{G}^{ext} = (c_{ijk} \cdot n) \cdot n$$

Second gradient theory

On a fixed wall ($V = 0$ is a constraint) no force has to be prescribed. However, a non trivial condition remains.

$$\mathcal{G}^{ext} = (c_{ijk} \cdot n) \cdot n$$

Mechanical interpretation :
- "Surface density of couple stress" for the tangent part,
- "Doubly normal double force" for the normal part.

What is the effect of such a condition on equilibrium or motion ?

A Cauchy-like construction of the theory

Now let study the reverse method : starting from the actions of the surounding medium on a part of the domain.
We first recall the classical Cauchy method. The hypotheses of Cauchy construction of stress are :

- H1) Actions can be represented by a surface density of forces F on the dividing boundary Σ.
- H2) F depends on the position x and orientation n of Σ : $F(x,n)$.
- H3) F is continuous with respect to x.
- H4) The action of F on a bounded domain is balanced by a bounded volume density of forces inside the domain (long range forces or inertia).

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- H2') $F(x, \Sigma)$ is uniformely bounded (Noll 1973).
- H3) F is continuous with respect to x.
- H4') For any smooth velocity field V, the power of mechanical boundary actions on a bounded volume is balanced by a volume density of power (power of long range forces or inertia) which is uniformely bounded.

Cauchy stress

Sketch of Cauchy's proof : consider

- a vanishing volume : *dependence with respect to x becomes negligible*. Volume quantities tends to zero faster than surface terms, hence *the action of F must be self balanced*.
- consider a tetrahedron with three faces with fixed direction *so F remains constant on them*) and the fourth face with variable direction n : *the balance implies (some computations . . .) that on this face $F(x,n)$ depends linearly on n.*

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Result : there exists a tensor $\sigma(x)$ such that $F(x,n) = \sigma(x) \cdot n$

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Result : there exists a tensor $\sigma(x)$ such that $F(x,n) = \sigma(x) \cdot n$
Assume that a line density $\mathcal{F}$ of forces is present along the edges. Cauchy theorem does not apply any more.

Edge forces

We assume that a line density $\mathcal{F}$ of forces is present along the edges and consider the domain

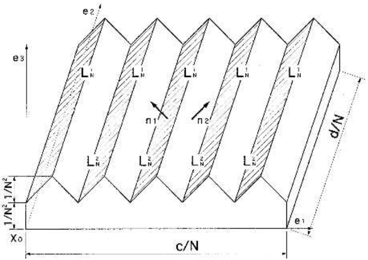


Figure 1. Grooved surface close to a plane

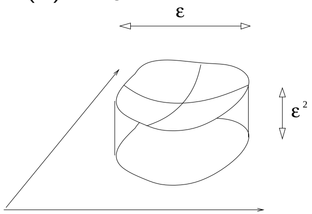
and the velocity field $V(x) = x_3 W$.

Theorem

The presence of surface and edge forces alone are impossible [F. Dell'Isola, P. S., 1997]: together with edge forces a new type of surface interaction must be present with a power of type $\int_\Sigma \mathcal{G} \cdot \frac{\partial V}{\partial n}$.

Edge forces

In the following we assume the presence of surface and edge forces plus an order one surface distribution $\mathcal{G}$.
Consider the velocity field $V(x) = x_3 W$ on the domain



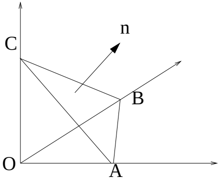
The actions $\mathcal{G}$ on top and bottom faces must balance each other. We get a result similar to Noll theorem:

Theorem

$$\mathcal{G} = \mathcal{G}(x, n).$$

Edge forces

Consider the velocity field $V(x) = ((x - A)\cdot n)W$ on the tetrahedron



3 fixed (n-independent) edge forces $\mathcal{F}(0, f_1)$ + 3 fixed $\mathcal{G}(0, e_i)$ actions must balance the n-dependent $\mathcal{G}$ action. Computation of lengths and areas give

Theorem

$$\mathcal{G}(0,n) = \mathcal{F}(0,f_1)(n.e_2)(n.e_3) + \mathcal{F}(0,f_2)(n.e_3)(n.e_1)$$
$$+\mathcal{F}(0,f_3)(n.e_1)(n.e_2) + \sum_{i=1}^{3} \mathcal{G}(0,e_i)(n.e_i)^2$$

Edge forces

Defining

$$c(x) = \frac{1}{2}\mathcal{F}(x,f_1)\otimes(e_2\otimes e_3+e_3\otimes e_2)+\frac{1}{2}\mathcal{F}(x,f_2)\otimes(e_3\otimes e_1+e_1\otimes e_3)+ +\frac{1}{2}\mathcal{F}(x,f_3)\otimes(e_1\otimes e_2+e_2\otimes e_1)+\sum_{i=1}^{3}\{G(x,e_i)\otimes e_i\otimes e_i\}$$

Theorem

$$\mathcal{G}(x,n) = (c(x).n).n$$

Edge forces

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Actions

$$\bar{F} := div^s(c\cdot n), \quad \bar{\mathcal{F}} := (c\cdot n)\cdot\nu, \quad \bar{\mathcal{G}} := (c_{ijk}\cdot n)\cdot n$$

are automatically balanced (by $\int_{\mathcal{D}} -c\cdot\nabla\nabla V$: see previous section)

Hence the differences $\tilde{F} = F - \bar{F}$, $\tilde{\mathcal{F}} = \mathcal{F} - \bar{\mathcal{F}}$, $\tilde{\mathcal{G}} = \mathcal{G} - \bar{\mathcal{G}} = 0$ are also balanced.

Edge forces

Our first theorem states that when $\mathcal{G} = 0$, then $\mathcal{F} = 0$. We get

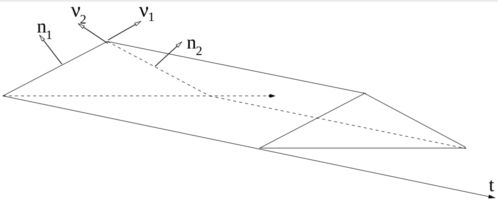
Theorem

$$\mathcal{F}(x,\Sigma) = (c(x)\cdot n_1)\cdot \nu_1 + (c(x)\cdot n_2)\cdot \nu_2$$

Moreover the “tilde actions” satisfy the Cauchy hypotheses: there exists a tensor σ such that $\tilde{F} = \sigma(x)\cdot n$

Theorem

$$F(x,\Sigma) = \sigma(x).n - div^s(c(x)\cdot n)$$



We recover the second gradient theory.

Second gradient material

A second gradient material is a material with constitutive equations involving the second gradient of the displacement.
To fix the ideas, let us consider the equilibrium of a linear elastic material:
In a domain Ω, un material has a displacement field u and the energy functionnal is $F(u)$. We assume that

- $F : \mathrm{L}^2(\Omega, \mathbb{R}^3) \to [0, +\infty]$, quadratic,
- $F(u) \geq \|u\|^2_{H^1}$ (coercive),
- F lower semi continuous (for the L^2 topology),
- f is a volume density of external forces ($\in \mathrm{L}^2$).

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- f is a volume density of external forces ($\in \mathrm{L}^2$).

An equilibrium solution exists which minimizes

$$\min_u \{F(u) - \int_\Omega f \cdot u\}$$

Second gradient material

Assume that F has the form

$$F(u) = \int_\Omega (\alpha \cdot \nabla u) \cdot \nabla u + (\beta \cdot \nabla\nabla u) \cdot \nabla\nabla u$$

if $u \in H^2$ and $u = 0$ on a no negligible part of the boundary, $F(u) = +\infty$ otherwise.

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Variational equation : for any admissible v

$$\int_\Omega (2\alpha \cdot \nabla u_0) \cdot \nabla v + (2\beta \cdot \nabla\nabla u_0) \cdot \nabla\nabla v - f \cdot v = 0$$

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Variational equation : for any admissible v

$$\int_\Omega \underbrace{(2\alpha \cdot \nabla u_0)}_{b} \cdot \nabla v + \underbrace{(2\beta \cdot \nabla\nabla u_0)}_{c} \cdot \nabla\nabla v - f \cdot v = 0$$

Integrating by parts

$$\int_\Omega (-div(b) + div(div(c)) - f) \cdot v + \int_{\partial\Omega} ((b - div(c)) \cdot n) \cdot v + ((c \cdot n) \cdot n) \cdot \frac{\partial v}{\partial n} + (c \cdot n) \cdot \nabla^s v = 0$$

$$\int_\Omega (\ldots) \cdot v + \int_{\partial\Omega} [(b - div(c)) \cdot n - div^s((c \cdot n) \cdot (Id - n \otimes n))] \cdot v + ((c \cdot n) \cdot n) \cdot \frac{\partial v}{\partial n} + \int_{\partial\partial\Omega} ((c \cdot n) \cdot \nu) \cdot v = 0$$

Second gradient material

Euler Equation :

$$\begin{aligned} div(b - div(c)) + f &= 0 \text{ on } \Omega, \\ (b - div(c)) \cdot n - div^s((c \cdot n) \cdot (Id - n \otimes n)) &= 0 \text{ on } (\partial\Omega)^{free}, \\ (c \cdot n) \cdot n &= 0 \text{ on } \partial\Omega, \\ [[(c \cdot n) \cdot \nu]] &= 0 \text{ on } (\partial\partial\Omega)^{free}, \end{aligned}$$

We are exactly in the framework of second gradient theory.

A mechanical error to avoid

Attempting to use first gradient theory (classical Cauchy theory) for describing second gradient material is an error.

First law of thermodynamics (variation of total energy)

$$\dot{E} + \dot{K} = \mathcal{P}^{ext} + Q_e$$

where $\dot{E}$, $\dot{K}$, Q_e are respectively the variations of internal and kinetic energies and the heat supply.

$$\dot{K} = \mathcal{P}^{int} + \mathcal{P}^{ext},$$

$$\dot{E} = -\mathcal{P}^{int} + Q_e \; .$$

Second law of thermodynamics states that the variation of entropy $\dot{S}$ is larger than the entropy supply Q_s:

$$\dot{S} \geq Q_s \; .$$

A mechanical error to avoid

Assume that E, S, $\mathcal{P}^{int}$ can be represented by volume densities e, s, p^{int}; and the supplies Q_e, Q_s by fluxes J_e, J_s then we get the Clausius-Duhem inequality

$$Tdiv(J_s) - div(J_e) - p^{int} + \rho\left(T\frac{d}{dt}(\frac{s}{\rho}) - \frac{d}{dt}(\frac{e}{\rho})\right) \geq 0$$

and in isothermal conditions

$$div(TJ_s - J_e) - p^{int} - \rho\frac{d}{dt}(\frac{\psi}{\rho}) \geq 0$$

($\psi = e - Ts$ is the volume free energy and T the absolute temperature).

A mechanical error to avoid

The thermodynamical “paradox” of second gradient materials lies in the incompatibility between Clausius-Duhem inequality

$$div(TJ_s - J_e) - p^{int} - \rho \frac{d}{dt}(\frac{\psi}{\rho}) \geq 0$$

and the three following assumptions:

(H_1) The free energy density ψ depends on the second gradient of the displacement.

(H_2) $p^{int} = -\sigma \cdot \nabla V$.

(H_3) $TJ_s = J_e$, consequence of
(i)J_e coincides with the heat flux ($J_e = q$) and
(ii) $J_s = q/T$.

Indeed, $div(TJ_s - J_e) = 0$ vanishes, $\frac{d}{dt}(\frac{\psi}{\rho})$ contains a term depending linearly on $\nabla\nabla V$ which cannot be balanced by p^{int}.

A mechanical error to avoid

One can revise (H_2) by using the second gradient theory, assuming that $\mathcal{P}^{int}$ has the form

$$p^{int} = -b \cdot \nabla V - c \cdot \nabla\nabla V = -\sigma : \nabla V - div(\nabla V^t : c)$$

One can revise (H_3) in two ways: either by introducing an "interstitial working" flux J^{int}, writing

$$J_e = q + J^{int}$$

or by writing

$$TJ_s = J_e - J^{int}.$$

All methods seem equivalent : the term $\nabla V^t : c$ plays the role of J^{int} and the difference is a question of nomenclature (what is called "power of internal forces").

This is not true : constitutive equations for b and c concern all admissible velocity fields, and not only the real one; the second gradient theory is stronger. Moreover it gives all the needed boundary conditions.

To make this point clear, we show in the next section the consequences of an application of extended thermodynamics to classical Cauchy continua.

A mechanical error generally avoided

Thermodynamics should give constraints on the possible constitutive laws and not give possiblities of getting over errors. In that sense “extended thermodynamics” is too much permissive..

A beginner’s error : “$div(\sigma)$ looks like a volume density of internal forces f^{int}, let us write the power of internal forces as

$$\mathcal{P}^{int} = \int_\Omega f^{int} \cdot V \, dv,$$

using so a zero-gradient theory”.

Is that a real error? Using extended-thermodynamics methods, introducing an extra flux J^{int}, we write

$$\rho \frac{d}{dt} V = f^{int}, \quad \rho \frac{d}{dt}\left(\frac{e}{\rho}\right) = f^{int} \cdot V - div(J^{int}) - div(q)$$

where f^{int} and J^{int} are given by suitable constitutive equations: for instance

$$f_i^{int} = p_{,i} + (\lambda + \mu) v_{j,ji} + \mu v_{i,jj}$$

$$J_j^{int} = (p + \lambda v_{j,j}) v_i + \mu v_j v_{j,i} + \mu v_j v_{i,j}$$

A mechanical error generally avoided

It is remarkable that

- this set of equations is totally equivalent to the classical set of equations (compressible Navier-Stokes)
- in such a presentation the notion of Cauchy stress tensor is not needed!
- the only (but important weakness) of the formulation is that boundary conditions *cannot* be written.
- Extended-thermodynamics is not able to detect the original mechanical error.

Claim : The same phenomenon occurs when extended-thermodynamics are used to describe a second gradient inside a first gradient theory.

Capillary fluid

A fluid with the following energy is clearly a second gradient material :

$$F(\rho) = \int_{\mathcal{D}} W(\rho) + \frac{\lambda}{2}\|\nabla\rho\|^2 + \int_{\mathcal{D}} m\rho$$

W is a Van der Waals potential with two minima at $\rho = \alpha$, $\rho = \beta$.
λ accounts for an extra energy due to strong gradient of density, m accounts for wall-fluid interactions. After some computations we get

$$\sigma = -pId - \lambda\nabla\rho\otimes\nabla\rho, \quad p = \rho\frac{\partial W}{\partial\rho} - W - \frac{\lambda}{2}\|\nabla\rho\|^2 - \lambda\rho\Delta\rho$$

$$c = -\lambda\rho Id\otimes\nabla\rho$$

So in a rigid container, we get the equation

$$\lambda\delta\rho = -\frac{\partial W}{\partial\rho} + Cste$$

with the boundary condition (corresponding to $\mathcal{G}$)

$$n\cdot\nabla\rho = -\frac{m}{\lambda}$$

Capillary fluid

Remarks

- edge forces can be computed on the edges of the container.
- surface forces depend on the curvature of the wall of the container.
- when λ tends to zero, the classical Laplace model of capillarity is recovered.
- the effect of the parameter m changes the contact angle.

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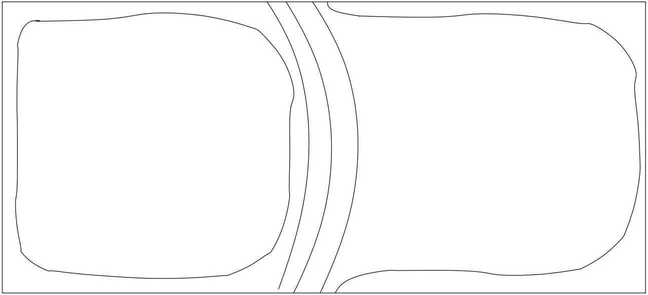

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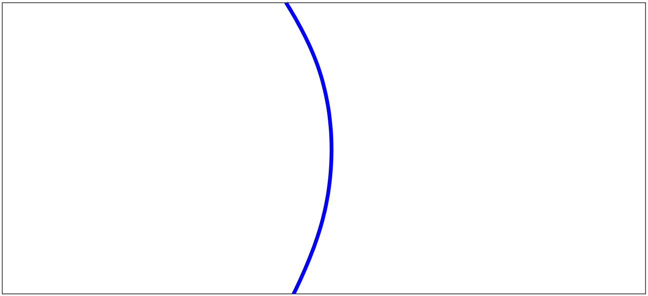
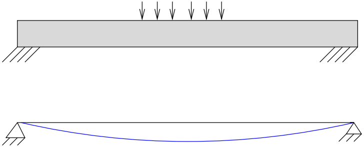


The beam in flexion

A 1D model obtained from classical 3D elasticity through an asymptotic process



- We denote u the transverse displacement.
- We use Dirichlet conditions $u(0) = u(\ell) = 0$.
- The elastic energy is $F(u) = \int_0^\ell k(u'')^2\,dx$. ($k$= flexural rigidity: a constitutive parameter)
- A transverse force density f is applied.

Equilibrium equation (Euler equation for minimization of total energy) is

$$div(div(2ku'')) - f = 0 \text{ on } [0, \ell]$$

Boundary conditions are $u(0) = 0$, $u(\ell) = 0$, $2ku''(0) = 0$, $2ku''(\ell) = 0$.

The beam in flexion

Remarks

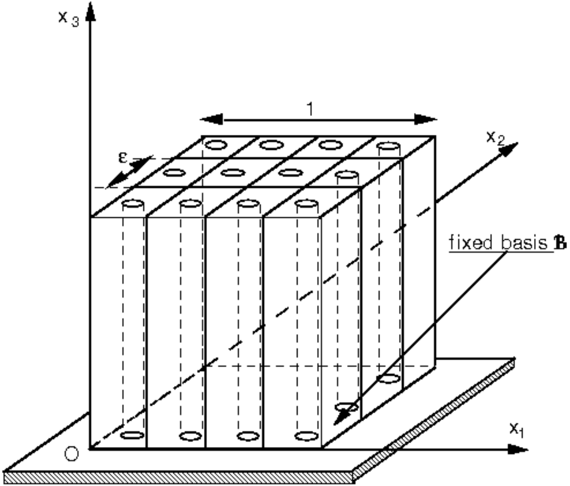
- It is not a good idea to write the equilibrium equation as

$$div(\sigma) - f = 0 \text{ with the constitutive law : } \sigma = (2ku'')'$$

- Setting $b = 0$, $c = 2ku''$, we recognize the second gradient theory applied to a second gradient material.
- u'' can be interpretated as a gradient of rotation and the extra boundary conditions as applied torques.
- an (asymptotic) link is made between classical and second gradient theory.

Is the dimension reduction fundamental here?

Homogenized network of beams



An isotropic linear elastic material with high contrast between matrix and fibers :

$$E_\varepsilon(u) = \int_{M_\varepsilon} [\frac{\lambda_0}{2}(Tr(e(u)))^2 + \mu_0 e(u)^2]\,dx + \int_{F_\varepsilon} [\frac{\lambda_\varepsilon}{2}(Tr(e(u)))^2 + \mu_\varepsilon e(u)^2]\,dx$$

if $u \in H^1$ and $u = 0$ on $\mathcal{B}$, $E_\varepsilon(u) = +\infty$ otherwise.

Homogenized network of beams

Geometric assumptions :

$$\lim_{\varepsilon\to 0} \frac{r_\varepsilon}{\varepsilon} = 0, \quad \lim_{\varepsilon\to 0} \varepsilon^2 \log(r_\varepsilon) = 0$$

Rigidity assumptions :

$$\lim_{\varepsilon\to 0} \frac{\mu_\varepsilon r_\varepsilon^4}{\varepsilon^2} = \mu_1 > 0, \quad \lim_{\varepsilon\to 0} \frac{\lambda_\varepsilon}{\mu_\varepsilon} = k$$

When ε tends to 0 we get the limit (homogenized) model

$$E_0(u) = \int_\Omega [\frac{\lambda_0}{2}(Tr(e(u)))^2 + \mu_0 e(u)^2]\,dx + \int_\Omega \frac{q}{2}\left[(\frac{\partial^2 u_1}{\partial x_3^2})^2 + (\frac{\partial^2 u_2}{\partial x_3^2})^2\right]\,dx$$

if $u \in H^1$, $\frac{\partial^2 u}{\partial x_3^2} \in L^2$, $u_3 = 0$ a.e. in Ω, $u = \frac{\partial u}{\partial x_3} = 0$ a.e. on $\mathcal{B}$. ($q = \frac{\pi}{4}\frac{3k+2}{k+1}\mu_1$)

Homogenized network of beams

Remarks
Here $u = \frac{\partial u}{\partial x_3} = 0$ a.e. on $\mathcal{B}$ is the dual of the $\mathcal{G}$ boundary condition.
A surface density of couples is applied on the basis.

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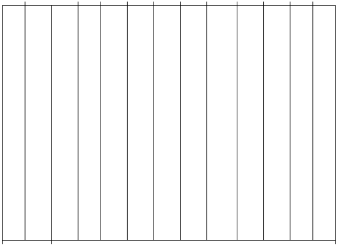

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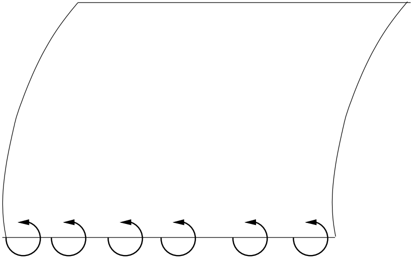


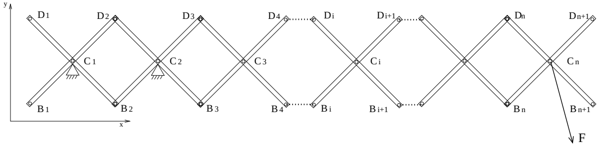
Homogenized network of beams

Remarks

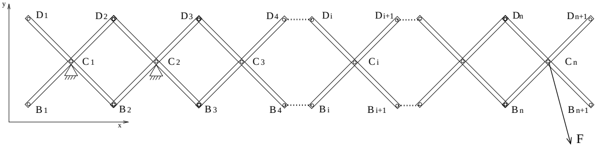
Here $u = \frac{\partial u}{\partial x_3} = 0$ a.e. on $\mathcal{B}$ is the dual of the $\mathcal{G}$ boundary condition.

A surface density of couples is no more applied on the basis.

Pantographic beam

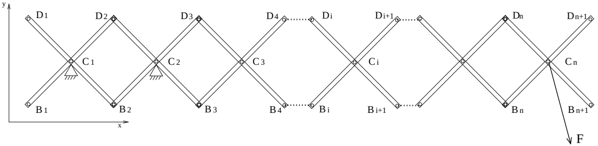


Pantographic beam



Transverse displacement u and axial displcement w.
Energy : $F(u,w) = \int_0^\ell k_v (u'')^2 + k_h (w'')^2 \, dx$.

Pantographic beam



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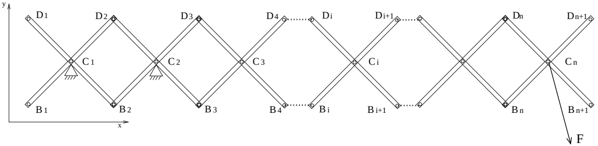
To the equilibrium equation for the flexion beam (for u) we add

$$div(div(2k_h w'')) - f_h = 0 \text{ on } [0,\ell].$$

Pantographic beam



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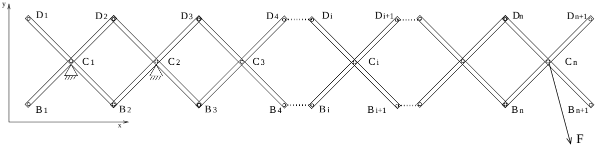

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Can a similisar 3D model be obtained by homogeneization ?

Closure of elasticity functionals

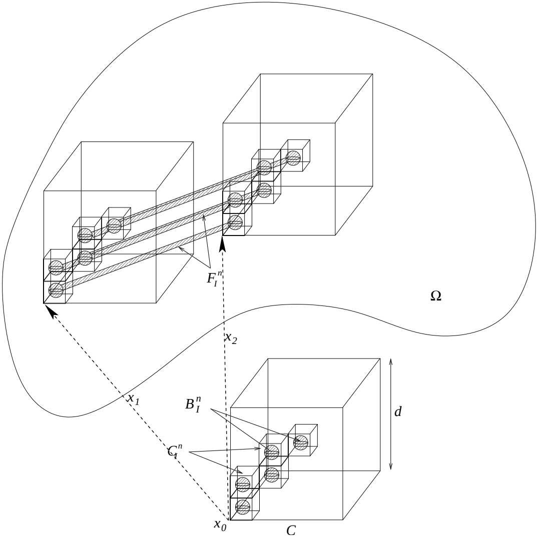
We prove : In dimension 3, every quadratic, non negative, l.s.c. and objective functionnal can be the energy of a material resulting from the homogeneization of a classical elastic medium

Remarks :

- We can moreover fix the Poisson coefficient of the classical elastic media we use.
- In particular negative Poisson coefficients are reachable.
- In particular complete second gradient elastic materials are reachable.

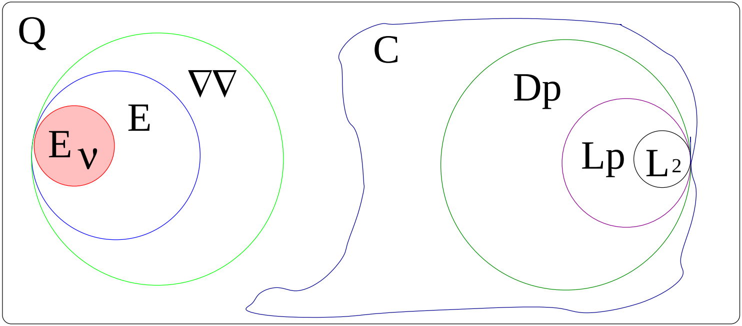
Closure of elasticity functionals

We start by proving that elementary non local interactions are possible.



Closure of elasticity functionals

Then we follow a long and abstract process



Conclusion

- Second gradient theory is now well written either in “D’Alembert” (Germain) or “Newton” (Cauchy) style.
- It has to be used when second gradient material are considered
- Second gradient materials are physically (and mathematically) usefull.
- Second gradient materials are intimely linked with classical Cauchy’s like material. They cannot be let outside of continuum mechanics.
- There are no new conceptual difficulties when considering higher gradient theories.