Lévy Walks and scaling in quenched disordered media

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Work in progress with

R. Livi - Firenze

D.Wiersma's group, (J. Bertolotti, P. Barthelemy, - K.Vynck, R.Savo here) - LENS Lab - Firenze

- A Lévy Walk for Light, or building tunable disordered materials for Lévy walks. The Levy Glass
- The Lévy Glass experiment: annealed and quenched Lévy Walks, and average experimental values
- Random (and deterministic) quenched 1-d Lévy models
- Transport and Diffusion on quenched 1d Lévy models: the effect of Averages over starting sites on asymptotics
- Future: higher dimensional samples and time resolved experiments

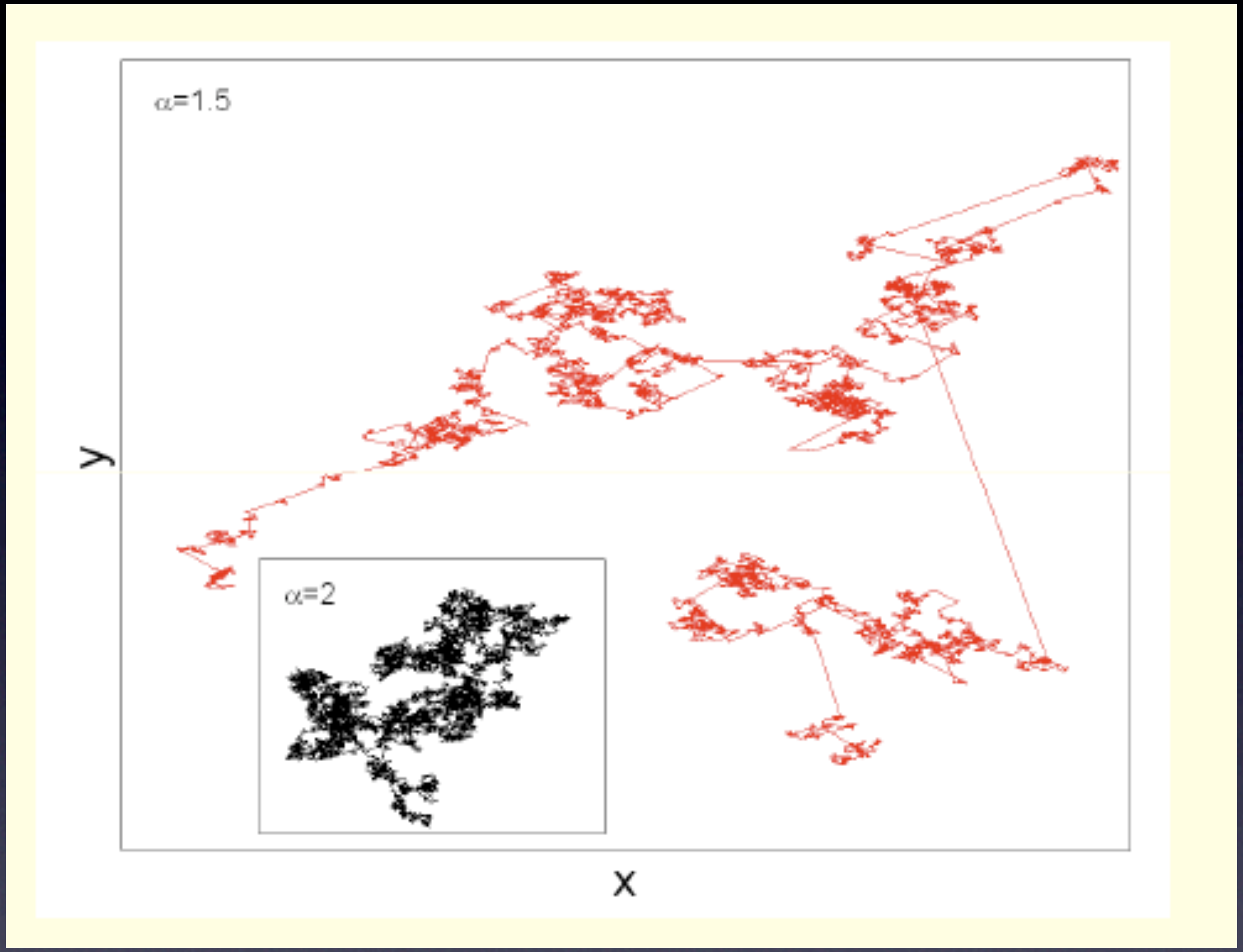
A Lévy Walk for light

An engineered disordered material where light performs a Lévy Walk-like motion, and superdiffuses.

Built in Florence - LENS Laboratory by D.Wiersma, J. Bertolotti and P. Barthelemy

In the lab, many experiments where light undergoes localization, Bloch oscillations, Hall effect in disordered samples.

Lévy walks: steps of length l in random direction.
The probability to take a long step of length l has a power law behavior, and long jumps can occur.



$$p(l) \sim \frac{1}{l^{\alpha+1}}$$ for large l

$$0 < \alpha < 2$$

Lévy walks and Lévy Fligths

- Lévy flights: each step takes a unit time

- Lévy walks: each step is covered at constant velocity, with time proportional to the step length l. A physical description.

Lévy Walks give rise to superdiffusive anomalous transport, in mean square displacement <r^2>:

$$\langle r^2(t) \rangle \sim t^\gamma \quad \gamma > 1$$

Lévy walks and Lévy Fligths

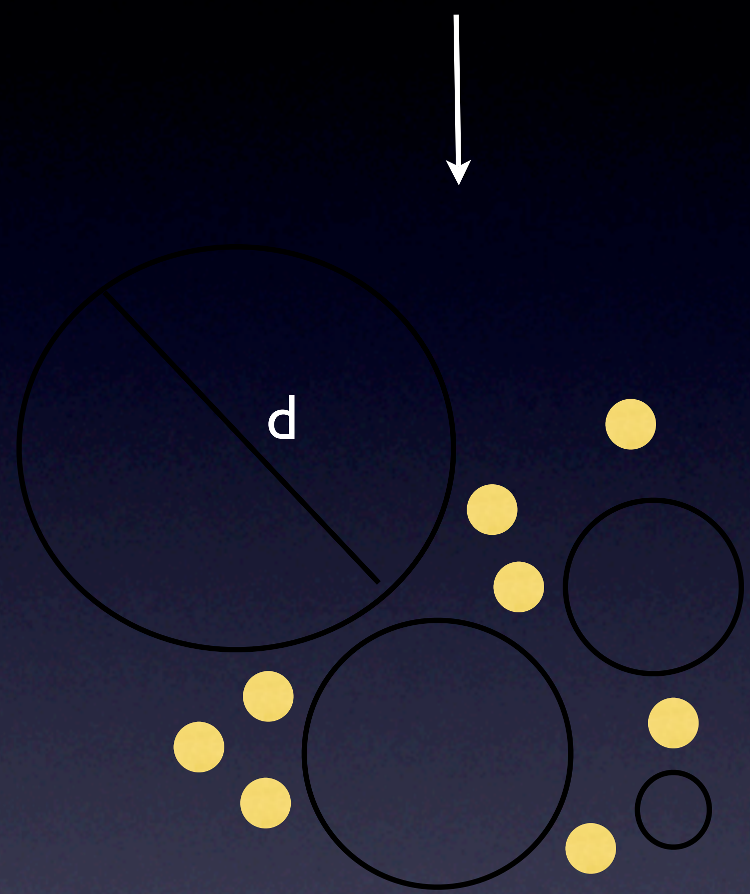
- Annealed Lévy walks: the lengths of the jumps are chosen randomly at each time step, i.e the steps are uncorrelated. Well known and studied.

- Quenched Lévy walks? Steps are correlated. How? From the geometry of a disordered material.
  Example:
  Light in Lévy like disordered materials!

How they built a Lévy like disordered materials at LENS

The Lévy Glass

Distribute the voids according to a power law, modifying the density of scatterers!



$$p(d) \sim \frac{1}{d^{\alpha+1}}$$

n.b. in 3d to have α one needs

$$p(d) \sim \frac{1}{d^{\alpha+2}}$$

The Lévy Glass

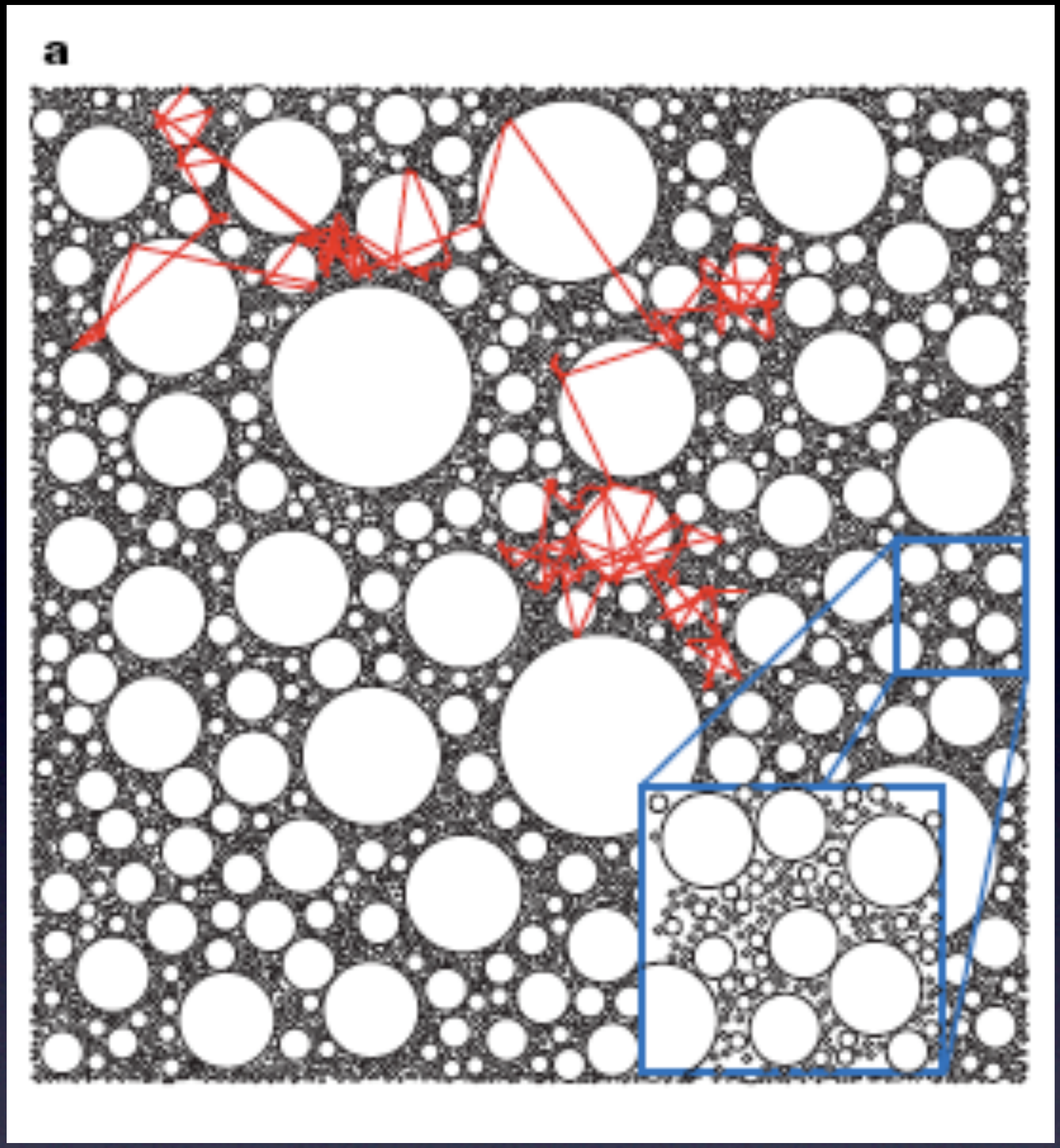
- A glass matrix (polymer now)

- Scattering medium (Ti O_2, Strong scatterers)

- Glass Spheres, with diameters distributed according to a Lévy tail, that do not scatter light (550-5μ m)

- Shake well, press and pack

- Quenched disorder! Correlated steps



The Lévy Glass

Measure of the transmission as a function of thickness L:
compatible with annealed Lèvy flight predictions (static measure)

$$T \sim L^{-\frac{\mu}{2}}$$

Evidence of Superdiffusion

D.Wiersma, J. Bertolotti and P.Barthelemy, Nature 2008
J. Bertolotti, K.Vynck, L- Pattelli, P. Barthelemy, S, Lepri and D.Wiersma, Adv Material 2010

-Superdiffusion vs transmission (for future time resolved experiments)?
What is the behavior of the mean square displacement?

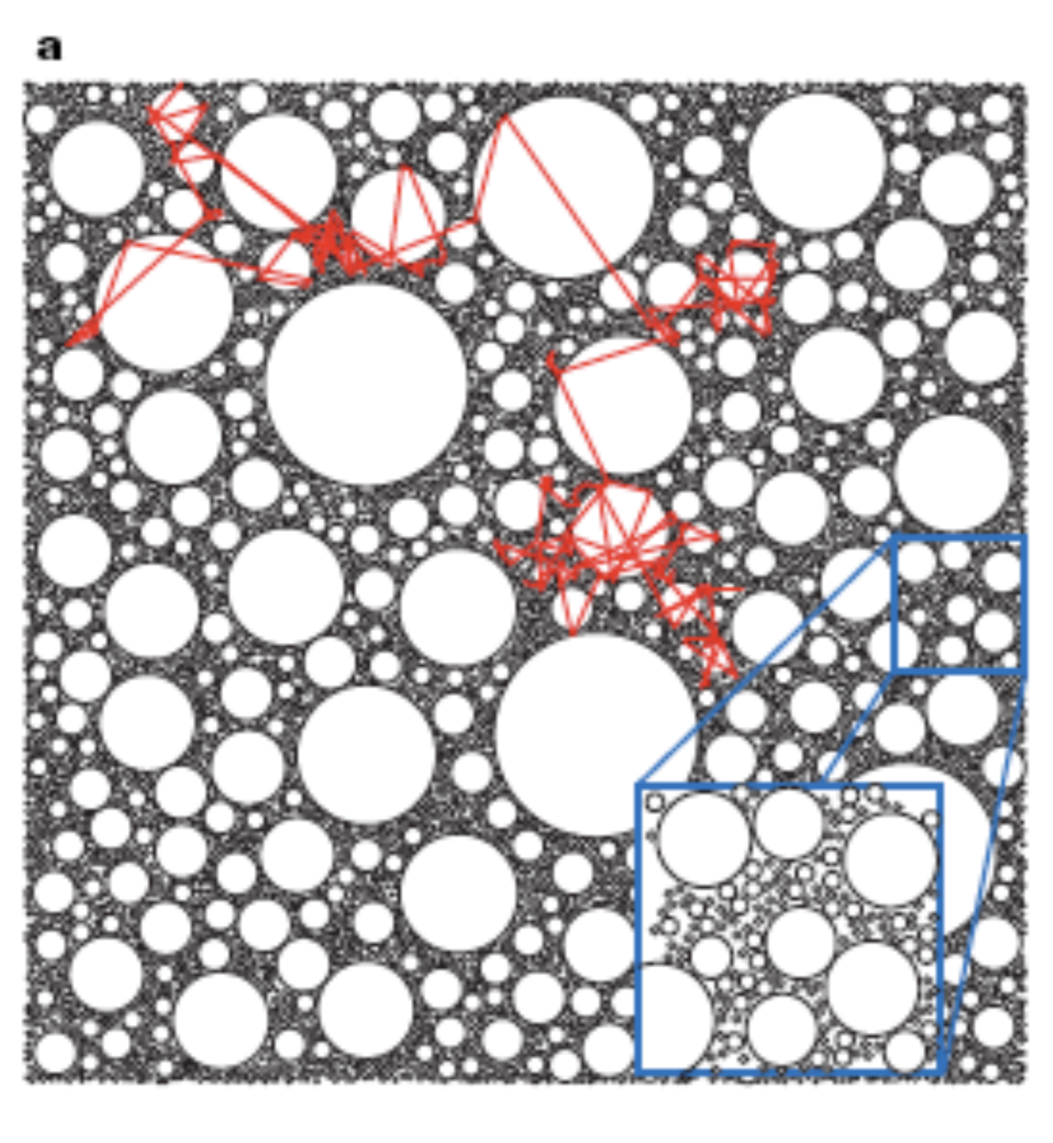
- Effects of the quenched disorder? This is a correlated Lévy walk, with the correlation induced by the topology of the sample.

The Lévy Glass

The process always starts with a scattering events.

Averages values should be calculated choosing a scattering site as a starting site.

First try: 1d models



Light in tunable Lévy-like disordered media:

- Testing Lévy like motion in tunable experiments
- Image reconstruction, medical imaging
- Experiments on light localization
- Random Lasers

One-dimensional models for the Lévy glass

• Simple models with quenched disorder (1d) where voids can be tuned by hand: self similar and random

• Control the dependence of the asymptotic laws on the starting point (averages) and the effects of long tails

• Relate transmission and diffusion through scaling laws

• An analytic estimate for exponents in the asymptotic region

• Difference between average and local measurements

• Different results under different average procedures

Annealed Lévy walks:

Annealed Lévy walks: the lengths of the jumps are chosen randomly at each time step, i.e the steps are uncorrelated

Known results:

$$\langle r^2(t)\rangle \approx \begin{cases} t^2 \ \mathbf{for}\ \ 0<\alpha<1 \\ t^{3-\alpha}\ \ \mathbf{for}\ \ 1<\alpha<2 \\ t\ \ \mathbf{for}\ \ \alpha>2 \end{cases} \qquad R(r)\approx\begin{cases} r^{\alpha/2}\mathbf{for}\ \ \alpha<2 \\ r\ \mathbf{for}\ \ \alpha>2 \end{cases}$$

$$R \sim \frac{1}{T}$$

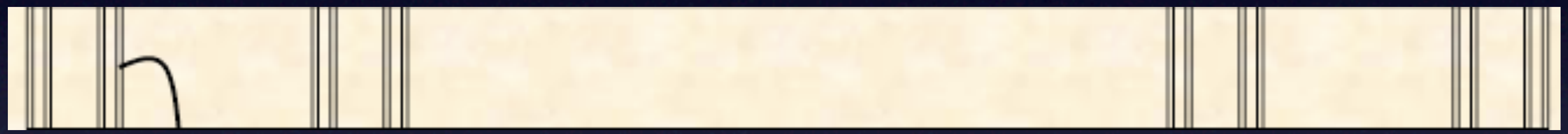
i.e. Geisel, Nierwetberg and Zacherl 1985, Zumofen and Klafter 1993

Quenched Lévy walks?

A deterministic model in 1d with quenched disorder: The cantor graphs

Deterministic Fractal: scatterers placed according to a Cantor set or Cantor- Smith Volterra set (A. Vezzani - poster)

scatterers: particle transmitted or reflected with prob 1/2

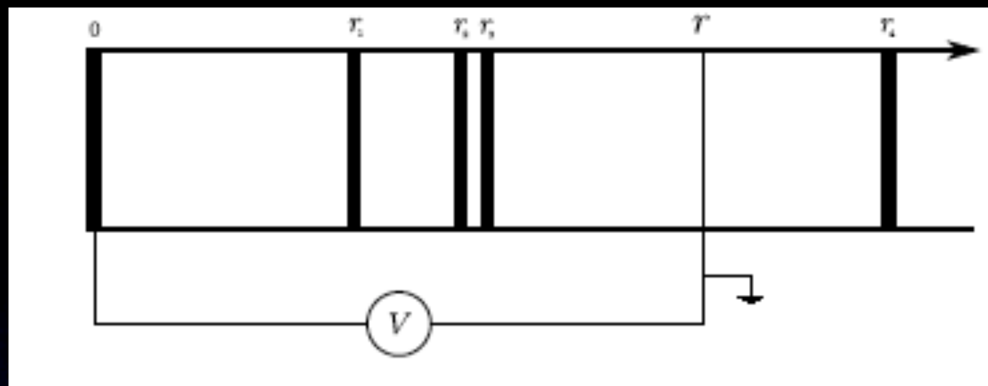


voids: ballistic motion with constant velocity

Analytic estimate for the scaling exponents

Local and average behavior (A. Vezzani - poster)

Closer to experiments:1d random models



$$p(r_{i+1}-r_i)=\frac{\alpha r_0^{\alpha}}{|r_{i+1}-r_i|^{1+\alpha}} \text{ if } |r_{i+1}-r_i|>r_0$$

$$p(r_{i+1}-r_i)=0 \textbf{ if } |r_{i+1}-r_i|>r_0 \textbf{ otherwise,}$$

Scatterers are placed in the positions $r_{i,}$ spaced according to a Lévy distribution with parameter α , r_0 sets the space scale

Two perspectives, treated independently:

Lévy walk model: the walker moves at constant velocity, hits a scatterer and it is transmitted or reflected with equal probability (Lévy-Lorenz gas) (Barkai, Fleurov, Klafter 2000)

Electric model: after a voltage is put between 0 to r, the resistance R(r) between the contacts is the number of scatterers between them (Beenakker, Groth, Akhmerov 2009)

The structure is given, so the disorder is quenched.

1d random quenched models: Known results

Different average procedures lead to different results:
if the Random Walker starts (contact are placed)

Beenakker et al (2009)

in any point of the structure		in a scattering point
$R(r) \approx \begin{cases} 0 \textbf{ for } \alpha < 1 \\ r \textbf{ for } \alpha > 1 \end{cases}$	Resistence	$R(r) \approx \begin{cases} r^{\alpha} \textbf{for } \alpha < 1 \\ r \textbf{ for } \alpha > 1 \end{cases}$
$\langle r^2(t) \rangle \approx \begin{cases} t^2 \textbf{ for } \alpha < 1 \\ t^{3-\alpha} \textbf{ for } 1 < \alpha < 2 \\ t \textbf{ for } \alpha > 2 \end{cases}$	Mean square displacemet	$\langle r^2(t) \rangle \approx \left\{ \; ? \right.$

The Lens experiment! Light enters in the sample with a scattering event.

this was an open point in the Lévy-Lorentz gas
(Barkai, Fleurov, Klafter 2000)

Tools:

- Mapping with the equivalent electric network problem

- Generalized scaling relations and the importance of tails

- The "single long jump" ansatz

Measurements: = particle starting point (contacts)

- Local (A. Vezzani - poster)

- Averaged over all points:

different results
for the asymptotic behavior

- Averaged only over scattering points:

The Random walk on Quenched Lévy graphs: the importance of averages

Local quantities: for a walker started on site i

$P_{ij}(t)$ Prob. of being on site j at time t

$P_{ii}(t)$ Return Probability at time t

$$\langle x_i^2 \rangle \equiv \sum_j x_{ij}^2 P_{ij}(t).$$

Mean square displacement (over RW realizations)

The Random walk on Quenched Lévy graphs: the importance of averages

Average quantities: On inhomogeneous graphs

$$\overline{P(t)} = \lim_{k\to\infty} \frac{1}{N_k} \sum_{i\in S_k} P_{ii}(t)$$

S_k sequence of graphs covering the infinite graph, $N_k = |S_k|$ or subsample of points

$$\overline{\langle x^2 \rangle} = \lim_{k\to\infty} \frac{1}{N_k} \sum_{i\in S_k} \langle x_i^2 \rangle$$

Averages and local quantities can behave differently on inhomogeneous graphs, even in the asymptotic region

R.B., D. Cassi, (2005) R.B., D. Cassi, A. Vezzani , in "Random Walks and geometry", V. Kaimanovich, K. Schmidt and W. Woess Eds, de Gruyter, Berlin (2004)

The Random walk on Quenched Lévy graphs

Asymptotic behavior at large times:

$$P_{ii}(t) \sim t^{-\frac{d_s}{2}}$$

$$\langle x_i^2 \rangle \sim t^{\gamma}$$

Anomalous diffusion:
Superdiffusion, subdiffusion, ballistic, normal

$$\overline{P(t)} \sim t^{-\frac{\bar{d}_s}{2}}$$

For SRW, weighted RW, RW with waiting probabilities
$\bar{d}_s$ is the spectral dimension of the graph

$$\overline{\langle x^2 \rangle} \sim t^{\bar{\gamma}}$$

average over scattering points?

Transmission: Random Walks and Electric Networks

Analogy between the RW master equation and the Kirchoff equations

$$-\sum_j L_{ji} V_j = \delta_{i0} - \delta_{in}$$

Unit current entering from i and going out from j, all links have unit resistance, V_i potential on site i

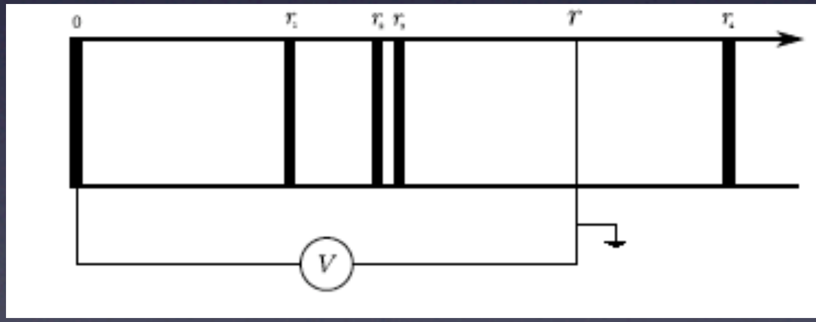
$$P_{0i}(t+1) - P_{0i}(t) = -\sum_j L_{ji} P_{0j}(t)/z_j + \delta_{i0}\delta_{t0}$$

RW starting from site i

With $L_{ij} = z_i \delta_{ij} - A_{ij}$

L Laplacian matrix of the graph
A adjacency matrix

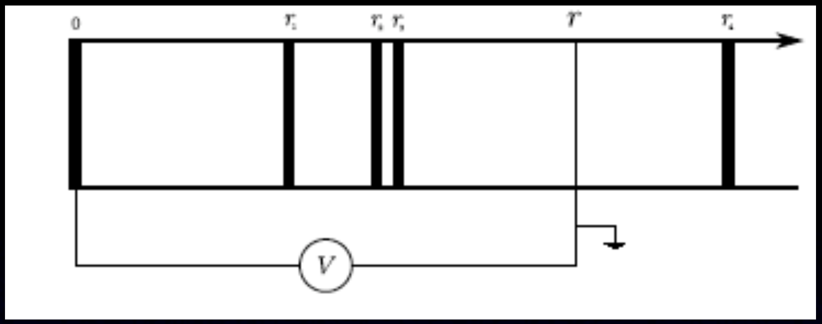
Fourier Trasform on time for P



$$\tilde{P}_{0i}(\omega)(e^{i\omega} - 1) = -\sum_j L_{ji} \tilde{P}_{0j}(\omega)/z_j + \delta_{i0}$$

Trasmission: Random Walks and Electric Networks

Then



$$V_i = \frac{1}{z_i} \lim_{\omega \to 0} (\tilde{P}_{0i}(\omega) - \tilde{P}_{ni}(\omega))$$

$$V_{0i} \equiv V(L) = R(L) \sim L^{\beta}$$

Resistance as function of the distance L between the two points 0 and L

The resistance is connected to the transmission at a distance L by

$$T(L) \sim R(L)^{-1} \sim L^{-\beta}$$

P.G. Doyle, J.L.Snell, Random Walks and Electric networks 2006

The scaling hypothesis and the Einstein relations

Assume that the most general scaling holds in 1d for the probability of being at a distance r. If l(t) is the scaling length, then:

$$P(r,t) = \ell^{-1}(t) f(r/\ell(t)) + g(r,t)$$ where $g(r,t)$ has zero measure

leading contribution to P

$$\lim_{t\to\infty} \int_0^{vt} |P(r,t) - \ell^{-1}(t) f(r/\ell(t))| dr = 0 \qquad \lim_{t\to\infty} \int |g(r,t)| dr = 0$$

Then:

1) from the normalization of P $\qquad \ell(t) \sim t^{d_s/2}$

2) from the expression for V_i $\qquad R(r) \sim r^{2/d_s - 1}.$

NB. d_s is the appropriate exponent, not necessarily the spectral dimension
$P(0,t) \sim t^{-d_s/2}$

Cates, 1985

The scaling hypothesis and the Einstein relations

Resistence: static problem! Much easier

Recalling the exact result for the resistance in averages over scattering points (Beenakker, Groth, Akhmerov 2009)

$$R(r) \approx \begin{cases} r^{\alpha} \textbf{ for } \alpha < 1 \\ r \textbf{ for } \alpha > 1 \end{cases} \longrightarrow \quad \ell(t) \sim \begin{cases} t^{\frac{1}{1+\alpha}} & \text{if } 0 < \alpha < 1 \\ t^{\frac{1}{2}} & \text{if } 1 \leq \alpha \end{cases}$$

The scaling length for P

The importance of long tails: anomalous effects

The standard behavior would be $\langle r^2(t)\rangle \sim \ell(t)^2$

But:

$$\langle r^2(t)\rangle = \int_0^{vt} \ell^{-1}(t) f(r/\ell(t)) r^2 dr + \int_0^{vt} g(r,t) r^2 dr.$$

$P(r,t)$ decays too slowly with r? (as in the annealed case) case 1

$g(r,t)$ decays too slowly with r and t? (as in Barkai et al) case 2

Here we have both cases, depending on alpha

The importance of long tails: how to estimate the anomalous effects. The "single long jump hypotesis"

Anomalous effects appears when r>> l(t). We can suppose that the walker reaches the distance r>> l(t) with a single long jump of length r, and the other scattering processes contribute until a distance l(t)!
Then:

$$P(r,t) \sim N(t)/r^{1+\alpha}$$

$N(t)$ Number of scatterers seen by the walker in a time t (but this is R!)

Prob. that a scatterer is followed by a jump of length r>>l(t)

$$N(t) \approx \begin{cases} \ell(t)^{\alpha} \text{ if } \alpha < 1 \\ \ell(t) \text{ if } \alpha > 1 \end{cases}$$

i.e.:

$$N(t) \approx \begin{cases} t^{\frac{\alpha}{1+\alpha}} \text{ if } \alpha < 1 \\ t^{\frac{1}{2}} \text{ if } \alpha > 1 \end{cases}$$

$$\boxed{R(r) \approx \begin{cases} r^{\alpha} \text{ for } \alpha < 1 \\ r \text{ for } \alpha > 1 \end{cases}}$$

Put all together and get:

$$\alpha < 1 \qquad r \gg \ell(t)$$

$$P(r,t) \sim t^{\alpha/1+\alpha} r^{-1-\alpha} = \frac{1}{\ell(t)}\left(\frac{r}{\ell(t)}\right)^{-1-\alpha}$$ case 1

$$\alpha > 1 \qquad r \gg \ell(t)$$

$$P(r,t) \sim t^{\alpha/2} r^{-1-\alpha} = \frac{t^{(1-\alpha)/2}}{\ell(t)}\left(\frac{r}{\ell(t)}\right)^{-1-\alpha}$$ case 2 $g(r,t)$

$g(r,t)$ provide a subleading contribution to P $\lim_{t\to\infty} \int_{\ell(t)}^{vt} g(r,t) = 0$

Mean square displacement:

$$\langle r^2(t)\rangle = \int P(r,t) r^2 dr \sim \underbrace{\ell(t)^2}_{r<\ell(t)} + \underbrace{\int_{\ell(t)}^{vt} N(t) r^{-1-\alpha} dr}_{r>\ell(t)}$$

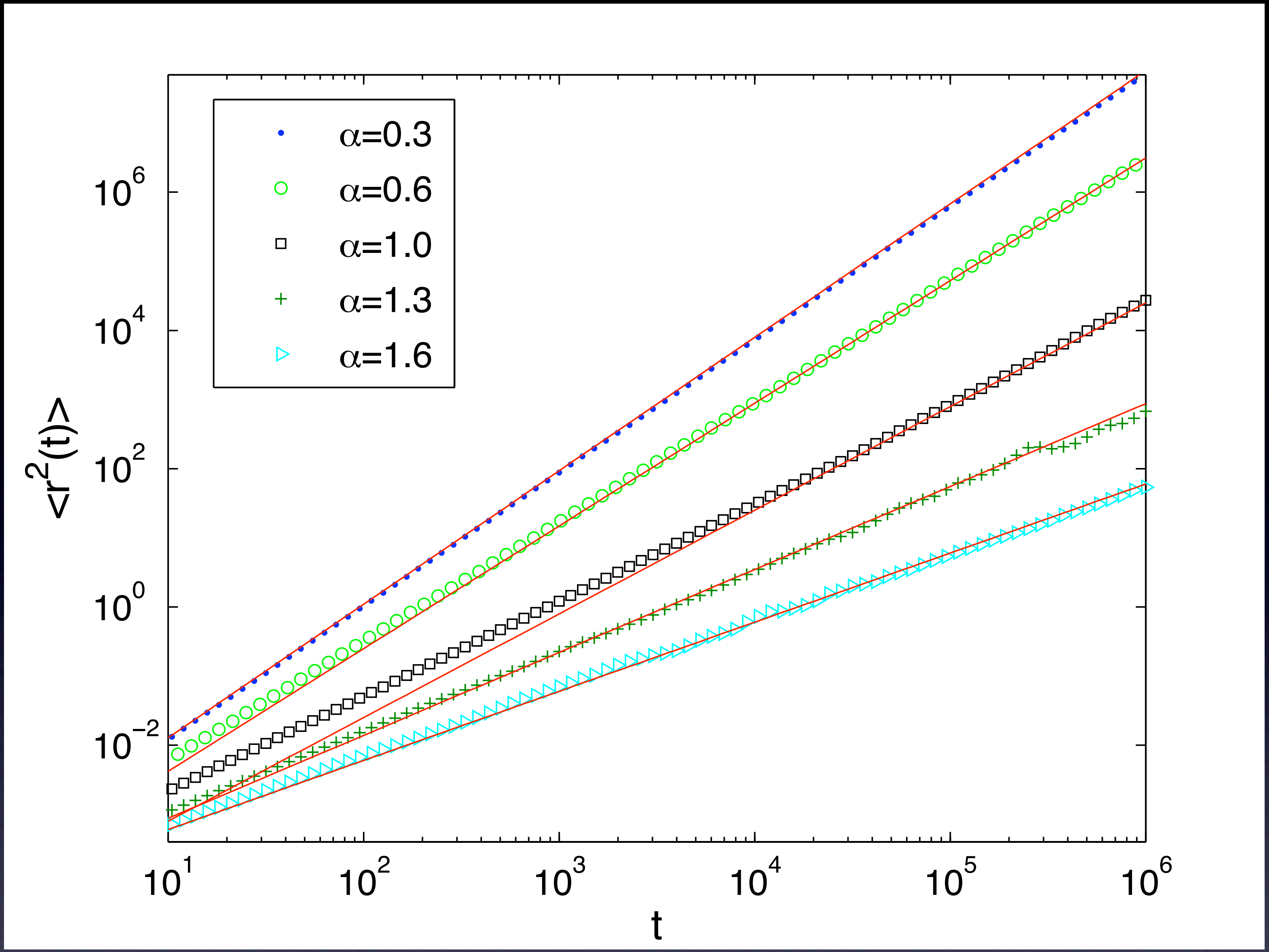
$$\boxed{\langle r^2(t)\rangle \sim \begin{cases} t^{\frac{2+2\alpha-\alpha^2}{1+\alpha}} & \text{if } 0<\alpha<1 \\ t^{\frac{5}{2}-\alpha} & \text{if } 1\le\alpha\le 3/2 \\ t & \text{if } 3/2<\alpha \end{cases}}$$

Moments of the mean square displacement:

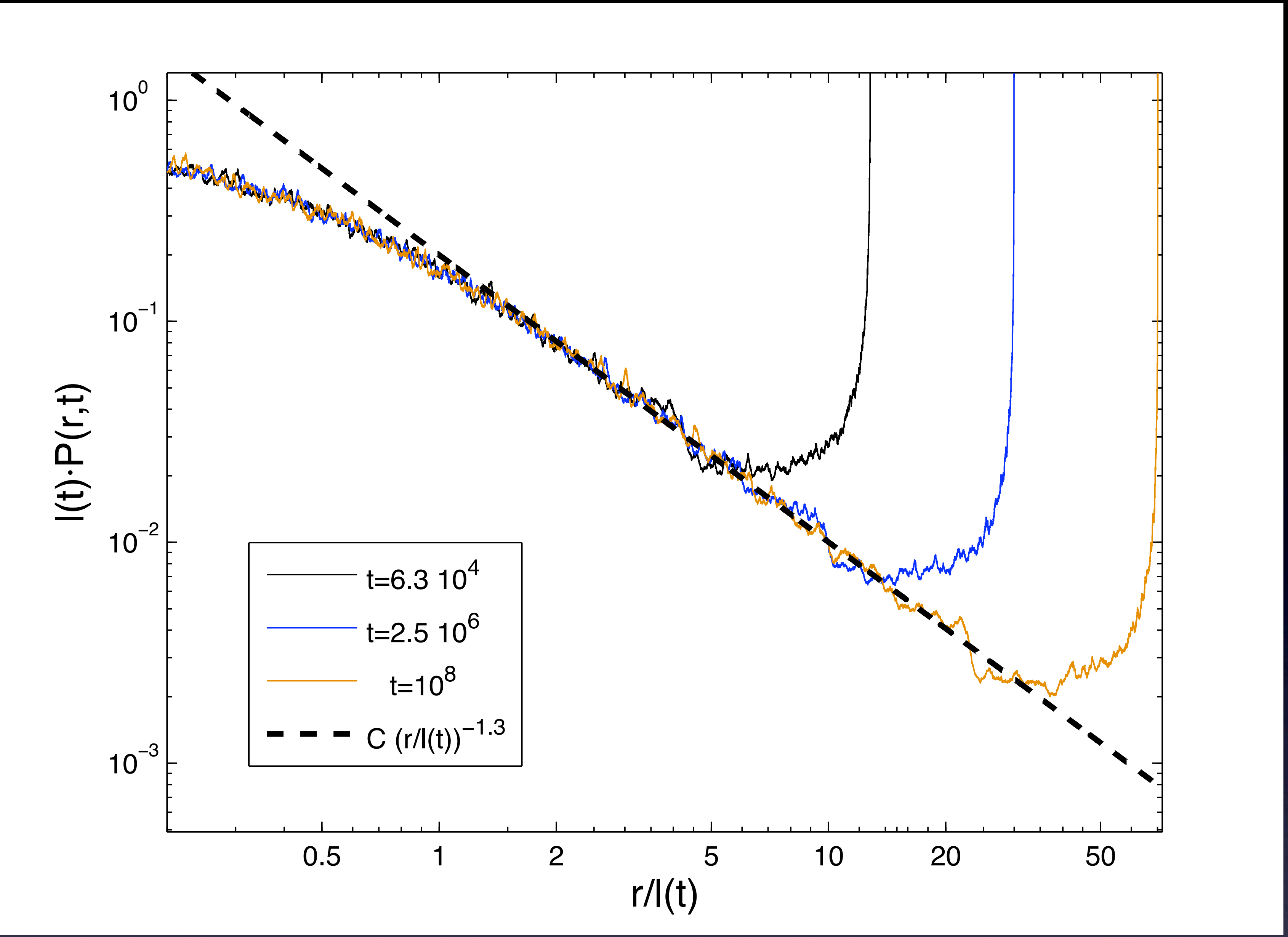
$$\langle r^p(t)\rangle \sim \begin{cases} t^{\frac{p}{1+\alpha}} \sim \ell(t)^p & \text{if } \alpha < 1,\ p < \alpha \\ t^{\frac{p(1+\alpha)-\alpha^2}{1+\alpha}} & \text{if } \alpha < 1,\ p > \alpha \\ t^{\frac{p}{2}} \sim \ell(t)^p & \text{if } \alpha > 1,\ p < 2\alpha - 1 \\ t^{\frac{1}{2}+p-\alpha} & \text{if } \alpha > 1,\ p > 2\alpha - 1 \end{cases}$$

$$q\,\nu(q) \simeq \begin{cases} \nu_1\, q & q < q_c \\ q - c & q > q_c. \end{cases}$$

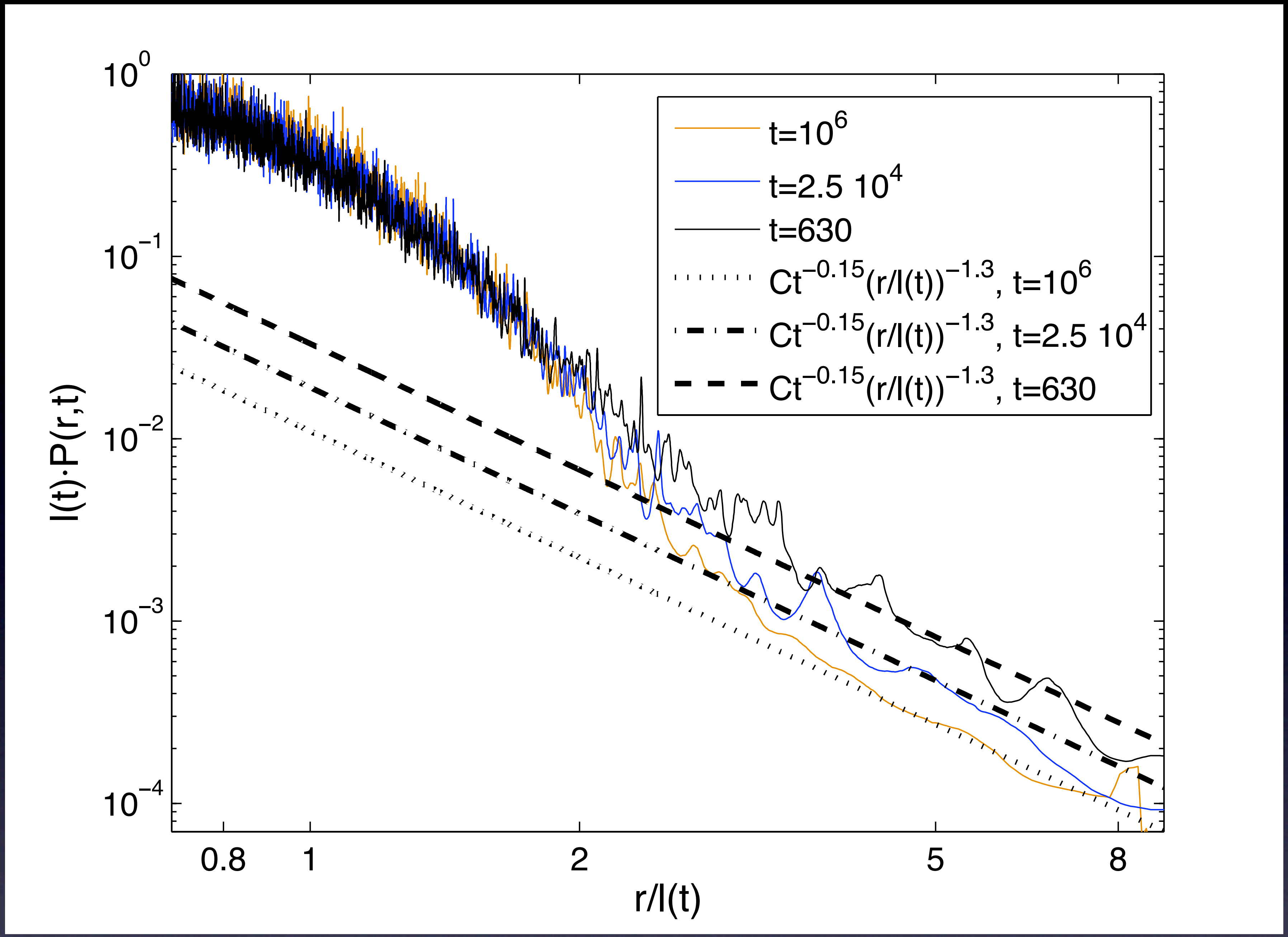
P. Castiglione, A. Mazzino, Muratore-Ginanneschi, A. Vulpiani 1999



α=0.3
α=0.6
α=1.0
α=1.3
α=1.6
<r^2(t)>
10^6
10^4
10^2
10^0
10^-2
10^1
10^2
10^3
10^4
10^5
10^6
t

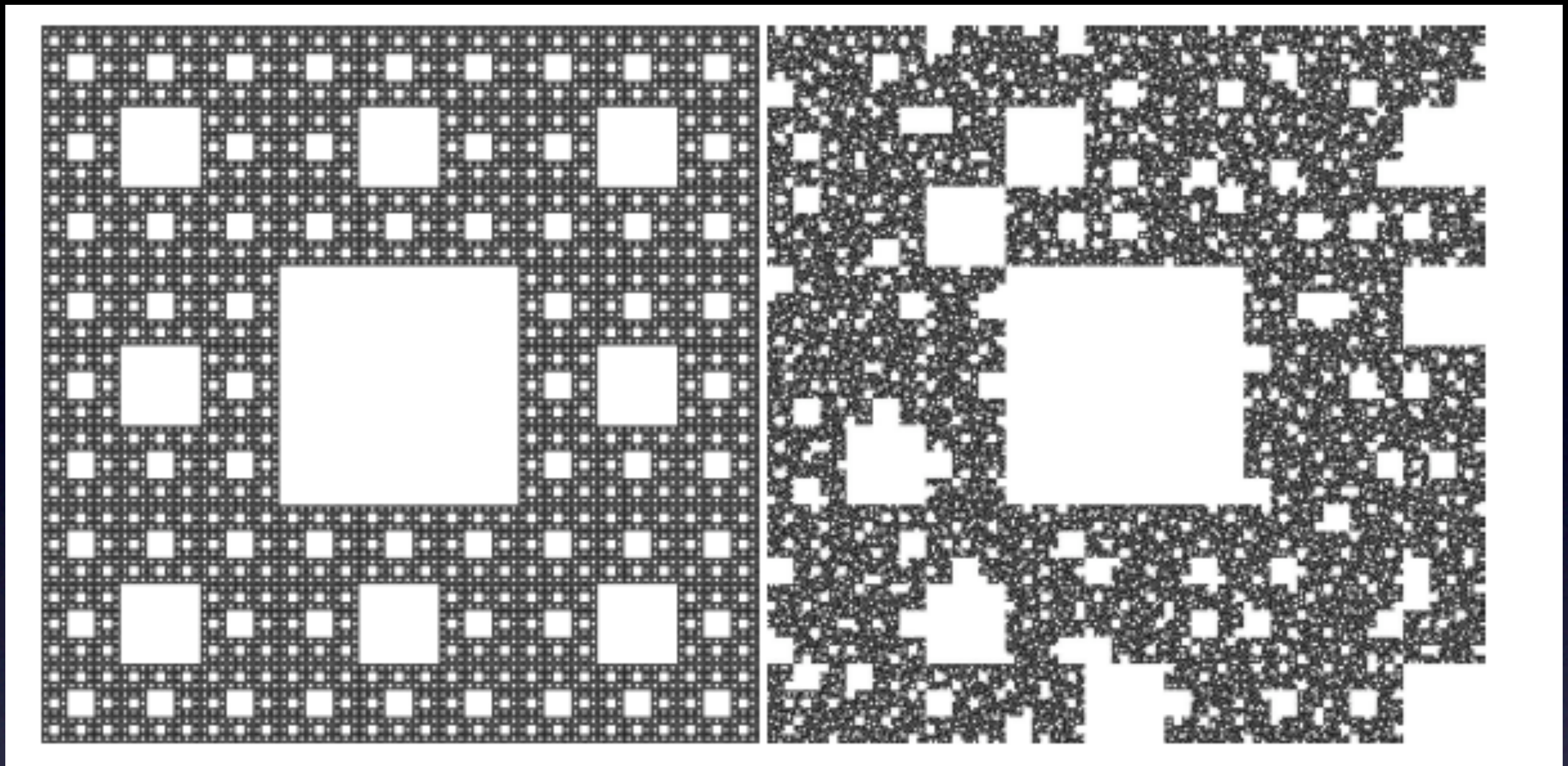


case 1: Montecarlo evaluation of the Prob density rescaled according to l(t) for $\alpha = 0.3$. For r>l(t) the behavior is as expected.



case 1: Montecarlo evaluation of the Prob density rescaled according to l(t) for $\alpha = 1.3$. For r>l(t) the behavior is as expected. The coefficient depends on time and vanishes as $t^{\frac{1-\alpha}{2}}$

2d: deterministic and random, alternative to numerical disk packing

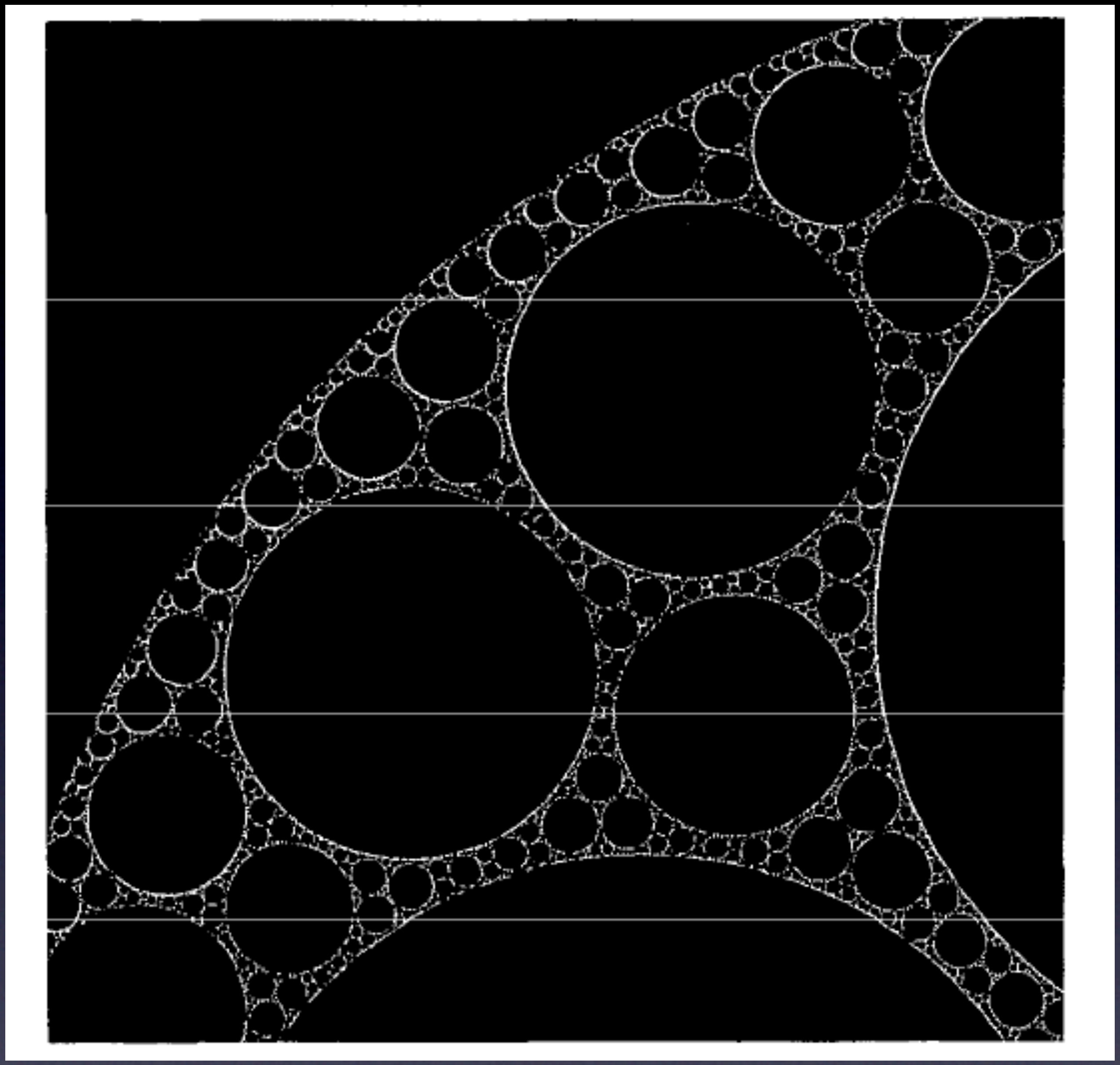


S. Lepri

Experiment: confined disks with directional/undirectional scattering particles, transmission and time resolved datas

Rigorous result for the single long jump ansatz?

An Apollonian packing of spheres



Bullet, Mantica 1992

R.B., L. Caniparoli, S. Lepri, A. Vezzani (2010), cond-mat 1005.3410
R.B., L. Caniparoli, A. Vezzani (2010)