Macroscopic fluctuations for non-equilibrium systems with mean-field interactions

Krzysztof Gawędzki
Rome, September 2013

" In our opinion, the mean field approach is very promising . . . It has a good heuristic power and there are still a lot of open problems in its framework ..."

M. Serva, **G. Paladin**, J. Raboanary in arXiv:cond-mat/9509005

• Nonequilibrium Stat. Mech. modeled by SDEs

$\nearrow$ **localized systems** — **colloidal particles**, **biomolecules**, **molecular motors**, ...

$\searrow$ **extended systems** — **stochastic lattice gasses** **zero range**, **SSEP**, **ASEP**, **ABC**, ...

- **Mean field systems** are somewhere in between :

$\nearrow$ for infinite systems they are described by a non−linear version of the small−system **Fokker−Planck** equation

$\searrow$ for large finite systems their fluctuations are described by theory developed for extended systems by the Rome school (**Bertini−DeSole−Gabrielli−Jona−Lasinio−Landim**)

- **Aim**: to add to the pool of few examples where the **macroscopic fluctuation theory** of the Rome group applies

- **Toolkit**: **large deviations** theory

- **Domain of application**: General non-equilibrium diffusions with mean-field coupling in dimension d:

$$\frac{dx_n}{dt} = X(t, x_n) + \frac{1}{N}\sum_{m=1}^{N} Y(t, x_n, x_m) + \sum_a X_a(t, x_n) \circ \underbrace{\eta_{na}(t)}_{\text{independent white noises}}$$

with $Y(x, y) = -Y(y, x)$ and $\circ$ for the **Stratonovich** convention

- **Based on**: joint work with **F. Bouchet** and **C. Nardini** in slow progress

- **Prototype model:** N planar rotators with angles θ_n and **mean field** coupling, undergoing **Langevin** dynamics

$$\frac{d\theta_n}{dt} = F - H \sin\theta_n - \frac{J}{N}\sum_{m=1}^{N} \sin(\theta_n - \theta_m) + \underbrace{\sqrt{2k_B T}\,\eta_n(t)}_{\text{independent white noises}}$$

Shinomoto-Kuramoto, Prog. Theor. Phys. **75** (1986),

. . . . . . ,

Giacomin-Pakdaman-Pellegrin-Poquet, SIAM J. Math. Anal. **44** (2012)

- Close cousin of the celebrated **Kuramoto** (1975) model for synchronization (with $F \to F_n$ and no noise)
- Models limit cycles of coupled nerve cells and their cooperative behaviors
- Close to models of depinning transition in disordered elastic media

$$\frac{d\theta_n}{dt} = F - H\,\sin\theta_n - \frac{J}{N}\sum_{m=1}^{N}\sin(\theta_n - \theta_m) + \sqrt{2k_BT}\,\eta_n(t)$$

- May be re-interpreted as a classical ferromagnetic **XY** model with a mean-field coupling of planar spins $\vec{S}_n$

  - $F = 0$ case (**equilibrium**):

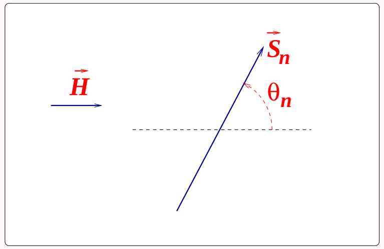
    in constant external magnetic field $\vec{H} = \big(H, 0\big)$

    $\vec{S}_n = S\big(\cos\theta_n, \sin\theta_n\big)$

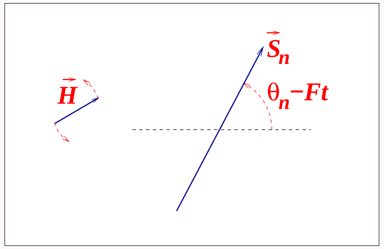
  - $F \neq 0$ case (**non-equilibrium**):

    in rotating external magnetic field $\vec{H} = H\big(\cos(Ft), -\sin(Ft)\big)$

    $\vec{S}_n = S\big(\cos(\theta_n - Ft), \sin(\theta_n - Ft)\big)$ (i.e. spins are viewed in the co−moving frame)



$F = 0$



$F \neq 0$

- **Macroscopic quantities of interest** in the general case

$$\frac{dx_n}{dt} = X(t,x_n) + \frac{1}{N}\sum_{m=1}^{N} Y(t,x_n,x_m) + \sum_a X_a(t,x_n)\circ\eta_{na}(t)$$

- **empirical density**

$$\rho_N(t,x) = \frac{1}{N}\sum_{n=1}^{N}\delta(x-x_n(t))$$

- **empirical current**

$$j_N(t,x) = \frac{1}{N}\sum_{n=1}^{N}\delta(x-x_n(t))\circ\frac{dx_n(t)}{dt}$$

- They are related to each other by the continuity equation:

$$\partial_t\rho_N + \nabla\cdot j_N = 0$$

- **Macroscopic fluctuation theory** applies to their **large deviations** at $N \gg O(1)$

- # Effective diffusion in the density space

- Substitution of the equation of motion for $\frac{dx_n(t)}{dt}$ and the passage to the **Itô** convention give:

$$j_N(t,x) = j_{\rho_N}(t,x) + \zeta_{\rho_N}(t,x)$$

where

$$j_\rho = \rho\big(\widehat{X} + Y * \rho\big) - D\nabla\rho \qquad \longleftarrow \quad \textbf{quadratic in} \quad \rho$$

with

$$\widehat{X} = X - \tfrac{1}{2}\sum_a \big(\nabla\cdot X_a\big)X_a\,, \qquad D = \tfrac{1}{2}\sum_a X_a \otimes X_a$$

$$(Y*\rho)(t,x) = \int Y(t,x,y)\,\rho(t,y)\,dy$$

and

$$\zeta_{\rho_N}(t,x) = \frac{1}{N}\sum_{n=1}^{N}\sum_a X_a(t,x)\,\delta\big(x - x_n(t)\big)\,\eta_{na}(t)$$

- Conditioned w.r.t. ρ_N, the noise $\zeta_{\rho_N}(t,x)$ has the same law as the **white noise** $\sqrt{2N^{-1}D(t,x)\,\rho_N(t,x)}\;\xi(t,x)$ where

$$\left\langle \xi^i(t,x)\,\xi^j(s,y)\right\rangle \;=\; \delta^{ij}\,\delta(t-s)\,\delta(x-y)$$

- Follows from the fact that for functionals

$$\Phi[\rho] \;=\; g\big(\textstyle\int \rho(x)h_1(x)dx,\ldots,\int\rho(x)h_k(x)dx\big)$$

the standard stochastic differential calculus gives

$$\frac{d}{dt}\left\langle \Phi[\rho_{Nt}]\right\rangle \;=\; \left\langle\,(\mathcal{L}_{Nt}\Phi)[\rho_{Nt}]\,\right\rangle$$

where

$$(\mathcal{L}_{Nt}\Phi)[\rho] \;=\; -\int \frac{\delta\Phi[\rho]}{\delta\rho(x)}\,\nabla\cdot j_\rho(t,x)\,dx$$

$$+\;\frac{1}{N}\int \frac{\delta^2\Phi[\rho]}{\delta\rho(x)\,\delta\rho(y)}\,\nabla_x\nabla_y\Big(D(t,x)\,\rho(t,x)\,\delta(x-y)\Big)\,dx\,dy$$

is the generator of the (formal) diffusion in the space of densities evolving according to the **Itô** **SDE**

$$\partial_t\rho \;+\; \nabla\cdot\big(j_\rho \;+\; \sqrt{2N^{-1}D\rho}\,\xi\big) \;=\; 0$$

- $N = \infty$ **closure**

  - When $N \to \infty$, the effective evolution equation for the **empirical density** reduces to **Nonlinear Fokker-Planck Equation** (**NFPE**)

    $$\partial_t \rho = -\nabla \cdot j_\rho = -\nabla \cdot \Big(\rho \big(\hat{X} + Y * \rho \big) - D \nabla \rho \Big)$$

    $\to$ a nonlinear dynamical system in the space of densities (autonomous or not)

  - If $Y = 0$ then the $N = \infty$ empirical density coincides with instantaneous **PDF** of identically distributed processes $x_n(t)$ and **NFPE** reduces to the linear **Fokker-Planck** equation for the latter

  - The $N = \infty$ **phase diagram** of an autonomous system with **mean-field** coupling is obtained by looking for stable stationary and periodic solutions of **NFPE** and the bifurcations

  - In principle, more complicated dynamical behaviors may also arise

- $N = \infty$ **phases of the rotator model**

  - Stationary solutions of **NFPE** satisfy:

$$\partial_\theta j_\rho(\theta) = \partial_\theta \Big(\rho(\theta) \Big(F - H \sin(\theta) - J \int_0^{2\pi} \sin(\theta - \vartheta)\, \rho(\vartheta)\, d\vartheta \Big) - k_B T\, \partial_\theta \rho(\theta) \Big)$$
$$= \partial_\theta \Big(\rho(\theta) \Big(F - (H + x_1) \sin\theta + x_2 \cos\theta \Big) - k_B T\, \partial_\theta \rho(\theta) \Big) = 0$$

with $\quad x_1 = J \int_0^{2\pi} \cos\vartheta\, \rho(\vartheta)\, d\vartheta\,, \qquad x_2 = J \int_0^{2\pi} \sin\vartheta\, \rho(\vartheta)$

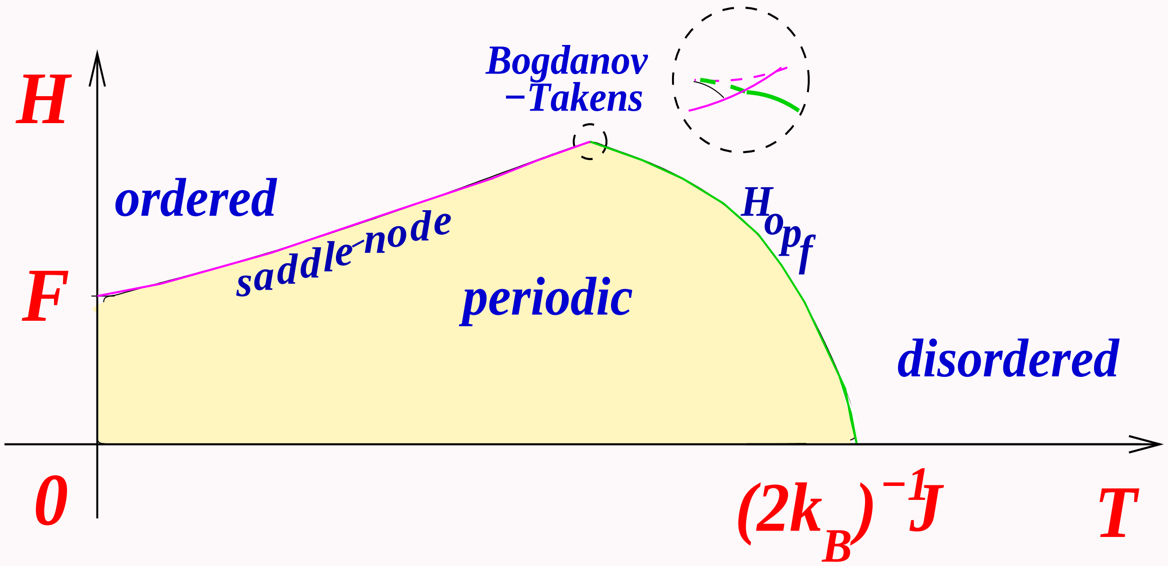
and the solution

$$\rho(\theta) = \frac{1}{Z}\, \mathrm{e}^{\frac{F\theta + (H + x_1)\cos\theta + x_2 \sin\theta}{k_B T}} \int_\theta^{\theta + 2\pi} \mathrm{e}^{-\frac{F\vartheta + (H + x_1)\cos\vartheta + x_2 \sin\vartheta}{k_B T}}\, d\vartheta$$

  - The coupled equations for 2 variables $x_{1,2}$ may be easily analyzed

  - The decoupled $J = 0$ solution corresponds to $x_{1,2} = 0$

- $N = \infty$ **phase diagram** for the **rotator model** for $F \neq 0$ (**Shi**-**Ku** 1984, **Sakaguchi**-**Shi**-**Ku** 1986, ...)



- For $H = 0$ the periodic phase coincides with the ordered low-temp. equilibrium phase viewed in the co-rotating phase

- When $F \searrow 0$ the periodic phase reduces to the equilibrium disordered phase at $H = +0$

- Global properties of the **NFPE** dynamics for the **rotator model** have been recently studied by **Giacomin** and collaborators

• Fluctuations for N large but finite

- Formally, domain of applications of the small-noise **Freidlin-Wentzell large deviations theory**

- In **Martin-Rose-Siggia** formalism, the joint **PDF** of empirical density and current profiles is

$$
\begin{aligned}
\Big\langle \delta\big[\rho-\rho_N\big]\,\delta\big[j-j_N\big]\Big\rangle \;&=\; \Big\langle \delta\big[\partial_t\rho+\nabla\cdot j\big]\,\delta\big[j-j_\rho-\zeta_\rho\big]\Big\rangle \\
=\; &\Big\langle \delta\big[\partial_t\rho+\nabla\cdot j\big]\int \mathrm{e}^{\,iN\int a\cdot(j-j_\rho-\zeta_\rho)}\,\mathcal{D}a\Big\rangle \\
=\; &\delta\big[\partial_t\rho+\nabla\cdot j\big]\int \mathrm{e}^{\,iN\int a\cdot(j-j_\rho)\,-\,N\int a\cdot\rho D\,a}\,\mathcal{D}a \\
\sim\; &\delta\big[\partial_t\rho+\nabla\cdot j\big]\,\mathrm{e}^{-\frac{1}{4}N\int(j-j_\rho)(\rho D)^{-1}(j-j_\rho)} \;\sim\; \mathrm{e}^{-N\,\mathcal{I}[\rho,j]}
\end{aligned}
$$

where the **rate function(al)**

$$
\mathcal{I}[\rho,j] \;=\; \begin{cases} \frac{1}{4}\int(j-j_\rho)(\rho D)^{-1}(j-j_\rho)\,dt\,dx & \text{if}\quad \partial_t\rho+\nabla\cdot j=0 \\ \infty & \text{otherwise} \end{cases}
$$

- Large-deviations rate function(al)s for empirical densities or empirical currents only

$$\Big\langle \delta[\varrho - \rho_N] \Big\rangle \underset{N\to\infty}{\sim} \mathrm{e}^{-N\mathcal{I}[\rho]} \qquad \Big\langle \delta[j - j_N] \Big\rangle \underset{N\to\infty}{\sim} \mathrm{e}^{-N\mathcal{I}[j]}$$

are obtained by the **contraction principle**

$$\mathcal{I}[\rho] = \min_j \mathcal{I}[\rho, j] = \frac{1}{4}\int \big(\partial_t\rho + \nabla\cdot j_\rho\big)\big(-\nabla\cdot\rho D\nabla\big)^{-1}\big(\partial_t\rho + \nabla\cdot j_\rho\big)\, dt\, dx$$

$$\mathcal{I}[j] = \min_\rho \mathcal{I}[\rho, j]$$ with appropriate boundary limiting conditions for ρ

- That empirical densities have dynamical large deviations with rate function given above was proven by **Dawson-Gartner** in 1987
- To our knowledge, the large deviations of currents for mean field models were not studied in math literature
- The formulae above have similar form as for the macroscopic density and current rate functions in stochastic lattice gases studied by the Rome group and **Derrida** with collaborators

- **Elements of the** (Roman) **macroscopic fluctuation theory**
  - **Instantaneous fluctuations of empirical densities**
    - Time t distribution of the empirical density

$$\mathcal{P}_t[\varrho] \;=\; \Big\langle \delta[\varrho - \rho_{Nt}] \Big\rangle \;\sim\; \mathrm{e}^{-N\mathcal{F}_t[\varrho]}$$

satisfies the functional equation $\partial_t \mathcal{P}_t \;=\; \mathcal{L}^{\dagger}_{Nt}\mathcal{P}_t$ which reduces for the large-deviations rate function $\mathcal{F}_t[\varrho]$ to the functional **Hamilton-Jacobi Equation** (**HJE**)

$$\partial_t \mathcal{F}_t[\varrho] + \int j_\rho \cdot \nabla \frac{\delta \mathcal{F}_t[\varrho]}{\delta \varrho} + \int \Big(\nabla \frac{\delta \mathcal{F}_t[\varrho]}{\delta \varrho}\Big) \cdot \rho\, D \Big(\nabla \frac{\delta \mathcal{F}_t[\varrho]}{\delta \varrho}\Big) \;=\; 0$$

    - In a **stationary state** the latter becomes the time-independent **HJE** for the rate function $\mathcal{F}[\varrho]$

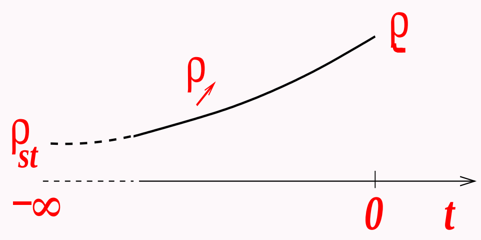
- **Relation between instantaneous and dynamical rate fcts**

  - By **contraction principle**

$$\mathcal{F}_t[\varrho] \;=\; \min_{\rho_t=\varrho} \Big(\mathcal{F}_{t_0}[\rho_{t_0}] \,+\, \mathcal{I}_{[t_0,t]}[\rho]\Big)$$

  - In the stationary state this reduces to

$$\mathcal{F}[\varrho] \;=\; \min_{\substack{\rho_{-\infty}=\rho_{st}\\ \rho_0=\varrho}} \; \mathcal{I}_{[-\infty,0]}[\rho]$$



    where ρ_{st} is the stable stationary solution of **NFPE** minimizing $\mathcal{F}[\varrho]$

  - The minimum on the right is attained on the most probable trajectory $\rho_{\nearrow}$ creating fluctuation ϱ from "vacuum" ρ_{st}

- **Time reversal**

  - One defines the **time-reversed current** $j'_\rho(t,x)$ by

$$j'^{\,*}_{\rho^*} \;=\; j_\rho \;+\; 2\,\rho D \nabla \frac{\delta \mathcal{F}_t[\rho_t]}{\delta \varrho}$$

where $\rho^*(t,x) = \rho(-t,x)$ and $j^*(t,x) = -j(-t,x)$ and the **time-reversed process** in the density space by **Itô** eqn.

$$\partial_t \rho' \;+\; \nabla \cdot \big(j'_{\rho'} \;+\; \sqrt{2N^{-1}D'\rho'}\,\xi\big) \;=\; 0$$

with $D'(t,x) = D(-t,x)$

  - **Fluctuation Relation**

$$\mathcal{I}_{[t_0,t_1]}[\rho,j] + \mathcal{F}_{t_0}[\rho_{t_0}] - \mathcal{F}_{t_1}[\rho_{t_1}] \;=\; \mathcal{I}'_{[-t_1,-t_0]}[\rho^*,j^*]$$

follows from the comparison of the direct and reversed rate functions and the **HJE** for $\mathcal{F}_t$

- **Generalized Onsager-Machlup Relation**

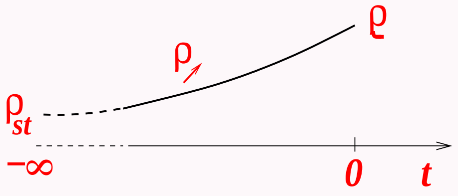
  - Upon minimizing over currents in a stationary state, Fluctuation Relation reduces to

$$\mathcal{I}_{[t_0,t_1]}[\rho] + \mathcal{F}[\rho_{t_0}] - \mathcal{F}[\rho_{t_1}] = \mathcal{I}'_{[-t_1,-t_0]}[\rho^*]$$

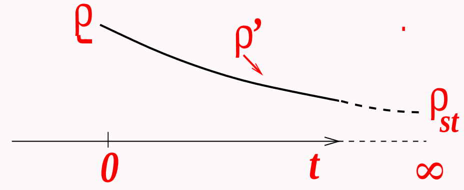
  - For $t_0 = -\infty$, $\rho_{t_0} = \rho_{st}$ and $t_1 = 0$, $\rho_{t_1} = \varrho$ the minimum of the **LHS** is attained on trajectory $\rho_{\nearrow}$ and is zero

  - It must be equal to the minimum of the **RHS** that is realized on trajectory $\rho'_{\searrow}$ that describes the decay of fluctuation ϱ to vacuum ρ_{st} and satisfies time-reversed **NFPE** $\partial_t \rho'_{\searrow} + \nabla \cdot j'_{\rho'_{\searrow}} = 0$

  - Hence the generalized **Onsager-Machlup** relation:



$$\rho_{\nearrow}(t,x) = \rho'_{\searrow}(-t,x)$$



- **Solutions for $\mathcal{F}_t$ in special cases**

  - For decoupled systems with $Y = 0$ and independent $x_n(0)$ all distributed with initial **PDF** ρ_0

    $$\mathcal{F}_t[\varrho] \;=\; \int \varrho(x)\,\ln\frac{\varrho(x)}{\rho_t(x)}\,dx \;\equiv\; k_B^{-1} S[\varrho\|\rho_t]$$

    where ρ_t solves the linear **FP** equation with initial condition ρ_0 (**Sanov Theorem**)

  - For stationary **equilibrium** evolutions with $\widehat{X}(x) = -M(x)\nabla U(x)$, $Y(x,y) = -M(x)(\nabla V)(x-y)$ and **diffusivity** and **mobility** matrices related by the **Einstein** relation $D(x) = k_B T\, M(x)$

    $$\mathcal{F}[\varrho] = \int \varrho(x)\left(\frac{1}{k_B T}\left(U(x) + \frac{1}{2}\int V(x,y)\,\rho(y)\,dy\right) + \ln\varrho(x)\right)dx + const.$$

    i.e. $k_B T\,\mathcal{F} = E - TS$ is the equilibrium mean-field **free energy**

- **Perturbative calculation of the non-equilibrium free energy** $\mathcal{F}[\varrho]$

  - $\mathcal{F}[\varrho]$ can be developed in a power series in mean-field coupling Y

    $$\mathcal{F}[\varrho] \;=\; \sum_{k=0}^{\infty} \mathcal{F}^k[\varrho]$$

    where $\mathcal{F}^0[\varrho] \;=\; k_B^{-1} S(\varrho \| \rho_{st}^0)$ with ρ_{st}^0 the stationary density of the decoupled model and $\mathcal{F}^k[\varrho]$ is of order k in Y

  - For $k \geq 1$, $\mathcal{F}^k$ is a non-local polynomial in ϱ of order $k+1$:

    $$\mathcal{F}^k[\varrho] \;=\; \tfrac{1}{(k+1)!} \int \phi^k(x_0, \ldots, x_k)\, \varrho(x_0) \cdots \varrho(x_k)\, dx_0 \cdots dx_k$$

    with ϕ^k are symmetric in the arguments

- Substituting the expansion for $\mathcal{F}$ into stationary **HJE** one obtains recursion relations

$$\int \varrho\, Q' \,\frac{\delta \mathcal{F}^k[\varrho]}{\delta \varrho} \;=\; \int \varrho \Big[\big(Y * \varrho\big) \cdot \nabla \frac{\delta \mathcal{F}^{k-1}[\varrho]}{\delta \varrho} + \sum_{l=1}^{k-1} \Big(\nabla \frac{\delta \mathcal{F}^{l}[\varrho]}{\delta \varrho}\Big) \cdot D \Big(\nabla \frac{\delta \mathcal{F}^{k-l}[\varrho]}{\delta \varrho}\Big) \Big]$$

, where $Q' = {\rho^0_{st}}^{-1} Q \rho^0_{st}$ and $Q = -\nabla \cdot \widehat{X} + \nabla \cdot D \nabla$ is the linear **FP** operator

- Kernels ϕ^k of $\mathcal{F}^k[\varrho]$ may be iteratively calculated from the above recursion in terms of a sum over tree diagrams

- For **rotator model**, the 1^{st} order correction has the form

$$k_B T\,\phi^1(\theta_0,\theta_1)$$

$$= \; J\big(1-\cos(\theta_0-\theta_1)\big) \qquad \longleftarrow \qquad \textbf{interaction energy contribution}$$

$$+\,\big(Q'\otimes Id + Id\otimes Q'\big)^{-1}\Big(\big(v^0_{st}\otimes 1 - 1\otimes v^0_{st}\big)J\sin(\cdot-\cdot)\Big)(\theta_0,\theta_1)$$

2−particle Green operator
the source of non−lacality

non−equilibrium contribution

where $v^0_{st}(\theta) = F - H\sin\theta - k_B T\,\partial_\theta \ln\rho^0_{st}(\theta)$ is the **mean local velocity** of the the decoupled system (vanishing for $F=0$)

- Another perturbative approach involves the expansion of $\mathcal{F}[\varrho]$ around its minimum ρ_{st} that is a stationary solution of **NFPE**

$$\mathcal{F}[\varrho] \;=\; \sum_{k=1}^{\infty} \widetilde{\mathcal{F}}^k[\widetilde{\varrho}]$$

where $\widetilde{\varrho} = \varrho - \rho_{st}$ and

$$\widetilde{\mathcal{F}}^k[\widetilde{\varrho}] \;=\; \tfrac{1}{(k+1)!} \int \widetilde{\phi}^k(x_0, \dots, x_n)\, \widetilde{\varrho}(x_0) \cdots \widetilde{\varrho}(x_k)\, dx_0 \cdots dx_k$$

with $\widetilde{\phi}^k$ symmetric in the arguments and fixed by demanding that $\int \widetilde{\phi}^k(x_0, x_1, \dots, x_k)\, dx_0 = 0$

- Substitution into the stationary **HJE** gives for $k > 1$ the recursion

$$\int \widetilde{\varrho}\, \Phi R \Phi^{-1} \frac{\delta \widetilde{\mathcal{F}}^k[\widetilde{\varrho}]}{\delta \widetilde{\varrho}} \;=\; \int \widetilde{\varrho} \Big[\big(Y * \widetilde{\varrho}\big) \cdot \nabla \frac{\delta \widetilde{\mathcal{F}}^{k-1}[\widetilde{\varrho}]}{\delta \widetilde{\varrho}}$$

$$+ \sum_{l=1}^{k-1} \Big(\nabla \frac{\delta \widetilde{\mathcal{F}}^l[\varrho]}{\delta \varrho}\Big) \cdot D \Big(\nabla \frac{\delta \widetilde{\mathcal{F}}^{k-l}[\varrho]}{\delta \varrho}\Big) \Big]$$

$$+ \sum_{l=2}^{k-1} \int \Big(\nabla \frac{\delta \widetilde{\mathcal{F}}^l[\widetilde{\varrho}]}{\delta \widetilde{\varrho}}\Big) \cdot \rho_{st} D \Big(\nabla \frac{\delta \widetilde{\mathcal{F}}^{k+1-l}[\widetilde{\varrho}]}{\delta \widetilde{\varrho}}\Big)$$

where R is the linearization of the nonlinear **Fokker-Planck** operator around ρ_{st} and

$$\left(\Phi\widetilde{\varrho}\right)(x) \;=\; \int \widetilde{\phi}^{1}(x,y)\,\widetilde{\varrho}(y)\,dy$$

solves the operator equation

$$R\Phi^{-1} + \Phi^{-1}R^{\dagger} \;=\; 2\nabla\cdot\rho D\nabla$$

(coming from the stochastic **Lyapunov** eqn.) and determines $\widetilde{\mathcal{F}}^{1}[\widetilde{\varrho}]$

- Kernels $\widetilde{\phi}^{k}$ of $\widetilde{\mathcal{F}}^{k}[\widetilde{\varrho}]$ may again be iteratively calculated from the above recursion in terms of a sum over tree diagrams

- For the **rotator model**, the 1^{st} expansion is better suited for the disordered phase whereas the 2^{nd} one for the ordered phase

- In both cases the leading corrections are accessible to numerical analysis that has been only started

• Large deviations for currents

- Following the Romans, one defines for time-independent current $\jmath(x)$

$$I_0[\jmath] \quad = \quad \lim_{\tau\to\infty} \frac{1}{\tau} \min_{\substack{\rho(t,x),\, j(t,x) \\ \jmath(x)=\frac{1}{\tau}\int_0^\tau j(t,x)\,dt}} \mathcal{I}_{[0,\tau]}[\rho, j]$$

- This is the rate function of large deviations for the temporal means $\jmath$ of current fluctuations

- In the stationary phase the minimum is attained on time independent (ρ, j) for $\jmath$ close to $j_{st} = j_{\rho_{st}}$ so that

$$I_0[\jmath] = \begin{cases} \min\limits_{\rho(x)} \frac{1}{4} \int (\jmath - j_\rho)(\rho D)^{-1}(\jmath - j_\rho)\, dx & \text{if} \quad \nabla\cdot\jmath = 0 \\ \infty & \text{otherwise} \end{cases}$$

but not necessarily for all $\jmath$

- In the periodic phase, it is more natural to look at

$$I_{\omega,\varphi}[\jmath] \quad = \quad \lim_{\tau\to\infty} \frac{1}{\tau} \min_{\substack{\rho(t,x),\, j(t,x) \\ \jmath(x)=\frac{1}{\tau}\int_0^\tau \sin(\omega t+\varphi)\, j(t,x)\, dt}} \mathcal{I}_{[0,\tau]}[\rho, j]$$

where ω is a multiple of the basic frequency

- **New phenomenon** that does not occur in equilibrium:

> At the 2nd order non-equilibrium phase transitions the covariance of temporal averages of current fluctuations around j_{st} on the scale $\frac{1}{N\tau}$ diverges in special directions

$\Rightarrow$ amplification of current fluctuations around such transitions

- In other words, the $N, \tau \to \infty$ variance of

$$\frac{\sum\limits_{n=1}^{N} \int\limits_{0}^{\tau} \delta j(t, x_n(t)) \circ dx_n(t) \; - \; \Big\langle \cdots \Big\rangle}{\sqrt{N\tau}}$$

(note the central-limit-like rescaling) diverges for some time-independent or periodic functions $\delta j(t,x)$ at such transitions

- A somewhat related enhancement of fluctuations at the saddle-node transition of the **rotator model** was observed numerically and analyzed in **Ohta-Sasa**, Phys. Rev. E **78**, 065101(R) (2008), see also **Iwata-Sasa**, Phys. Rev. E. **82**, 011127 (2010)

- The inverse covariance of the current fluctuations is extracted by expanding the rate functional $\mathcal{I}[\rho, j] = \frac{1}{4} \int (j - j_\rho)(\rho D)^{-1}(j - j_\rho)$ to the 2^{nd} order around (j_{st}, ρ_{st}):

$$\mathcal{I}[\rho_{st} + \delta\rho, j_{st} + \delta j] = \frac{1}{4}\int(\delta j - S\delta\rho)(\rho_{st} D)^{-1}(\delta j - S\delta\rho)$$

where $S(x.y) = \frac{\delta j_{\rho_{st}}(x)}{\delta\rho(y)}$

- The linearized **Fokker**- **Planck** operator is $R = -\nabla \cdot S$

- At critical points corresponding to a saddle-node or a pitchfork bifurcations, R has a zero mode $\delta\rho_0(x)$ and then for $\big(\delta\rho(x), \delta j(x)\big) = \big(\delta\rho_0(x), (S\delta\rho_0)(x)\big)$

$$\delta j - S\delta\rho = 0$$

so that $\mathcal{I}[\rho_{st} + \delta\rho, j_{st} + \delta j]$, and consequently $I_0[j_{st} + \delta j]$, vanish to the 2^{nd} order on such a perturbation

- At critical points corresponding to a **Hopf** bifurcation, R has a pair of complex conjugate modes $\delta\rho_0(x)$, $\overline{\delta\rho_0(x)}$ with eigenvalues $\pm i\omega$ and then for $(\delta\rho, \delta j) = \mathrm{Re}\,(\mathrm{e}^{i\omega(t+t_0)}\delta\rho_0,\, \mathrm{e}^{i\omega(t+t_0)}S\delta\rho_0)$

$$\delta j - S\delta\rho \;=\; 0$$

  and again $\mathcal{I}[\rho_{st} + \delta\rho, j_{st} + \delta j]$, and consequently $I_{\omega,\varphi}[\mathrm{Re}\,\mathrm{e}^{i\psi}S\delta\rho_0]$ for any phase ψ vanish to the 2^{nd} order

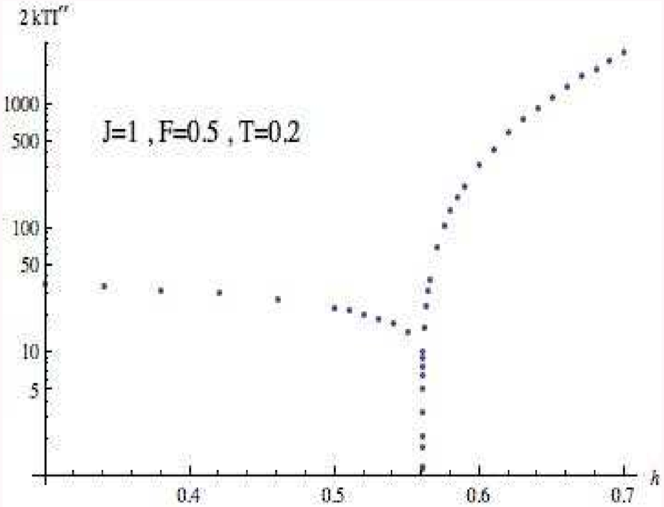
- Note that in both cases the constraint $\partial_t\delta\rho + \nabla\cdot\delta j = 0$ is satisfied

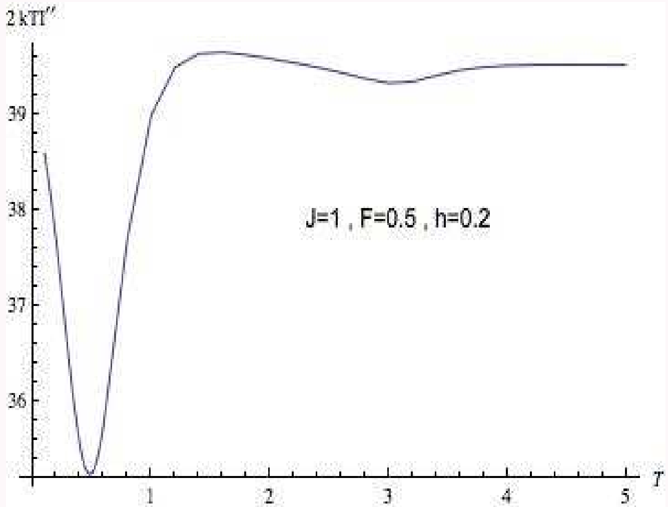
- Vanishing of $\mathcal{I}$, I_0 or $I_{\omega,\varphi}$ to the 2^{nd} order around j_{st} means that the covariance of current fluctuations in the corresponding directions diverges on the central-limit scale $\frac{1}{N\tau}$

- The reason is that such fluctuations are realized in $N = \infty$ dynamics

- In equilibrium, R cannot have non-zero imaginary eigenvalues and for its zero modes $\delta\rho_0$, one also has $S\delta\rho_0 = 0$, unlike in nonequilibrium where $\delta j = S\delta\rho_0$ represents a non-trivial current fluctuation

Example of the **rotator model** for $J = 1, \; F = 0.5$





The inverse covariance $2k_B T I_0''[j_{st}]$ as a function of magnetic field h (left, with log-lin scale) and temperature T (right, with lin-lin scale)

- The left figure illustrates the vanishing of $I_0''[j_{st}]$ at the saddle-node bifurcation for $h = h_{cr} \approx 0.56$ (the points for $h < h_{cr}$ correspond to an unstable stationary solution within the periodic phase)

- The right figure shows the non-zero behavior of $I_0''[j_{st}]$ near the Hopf bifurcation at $T = T_{cr} \approx 0.5$ (again, the $T < T_c$ curve corresponds to a stationary solution that is unstable within the periodic phase)

Conclusions and open problems

- Diffusions with mean field coupling are described for $N = \infty$ by **NFPE** and may exhibit interesting phase diagrams with dynamical transitions.
- Large deviations of empirical densities and currents for large but finite N are described in such models by rate functionals similar as for stochastic lattice gases, leading in the macroscopic fluctuation theory
- The non-equilibrium free energy satisfies a functional **Hamilton-Jacobi** eq. whose solutions may be studied in perturbation theory
- The covariance of current fluctuations diverges in specific directions at the 2^{nd} order transition points of such systems, unlike in equilibrium
- The analysis of concrete systems, like the **rotator model**, may be done by combining analytical and numerical arguments and requires more work, in particular on large deviations of currents
- Similar methods should apply to underdamped diffusions with mean-field coupling leading at $N = \infty$ to **Vlasov- Fokker- Planck** eq. We hope also to apply them to randomly forced 2D **Navier- Stokes** eqns.