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Motion of a granular particle on a rough line

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Abstract. – We discuss a new model of ideal granular gas consisting of a particle bouncing inelastically along a rough inclined plane. Assuming a velocity-dependent inelastic interaction between the surface and the falling object we study the dynamical phase diagram which consists of three different phases: an accelerated motion, a stopping phase and a phase where the velocity fluctuates about a constant value. We analyze the statistical properties of the steady velocity regime and find that the velocity distribution is characterized by power law tails with a nonuniversal exponent β which depends on the nature of the surface. An explanation for this phenomenon is presented and its relation with random multiplicative processes expounded.

Granular flows represent an active research field due to their scientific and technological importance: not only they occur in nature, often contributing to determine the landscape of deserts, mountains, beaches, dunes, etc., but are also relevant to many industrial and manufacturing processes.

In the present letter we study a special case of granular flow, namely the motion of a single granular particle down an inclined rough plane. This can be thought as the most elementary process contributing to the complex dynamics of surface flows and is relevant to determine the conditions under which a flow is steady. Such a regime arises in the presence of a driving field, gravity in this case, and of inelasticity in the collision process between the surface and the falling object. The model is inspired to recent laboratory experiments [1,2] and numerical simulations of the motion of a bead down a rough surface. Motivated by recent studies [3] which revealed that in the presence of inelastic collisions the velocity distribution of a driven granular system deviates from a Maxwell-Boltzmann distribution, we examine the following important issue: which are the statistical properties of the velocity in the steady phase?

We shall consider the dynamical properties of a single bead bouncing inelastically on an inclined surface [4–7]. With respect to previous studies we assume that the surface is stochastically rough, a situation commonly encountered in nature where for example it may correspond to a stone falling down a scree.

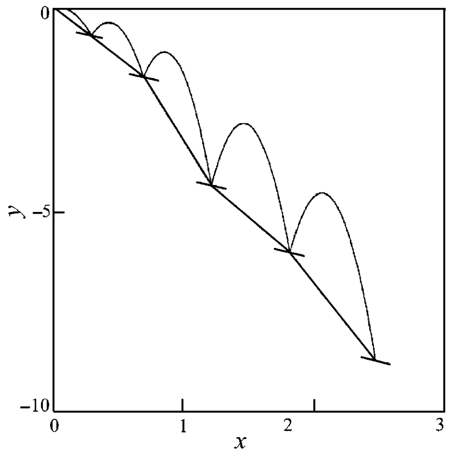

Fig. 1

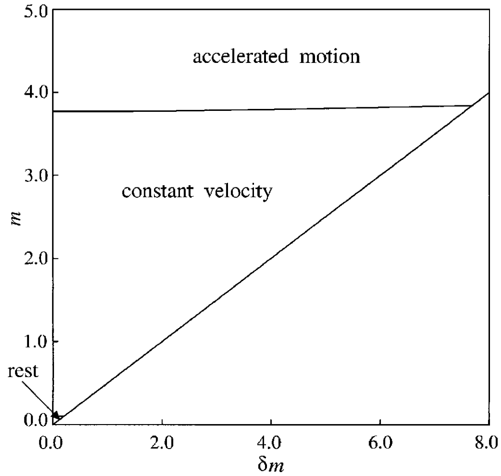

Fig. 2

Fig. 1 – Geometry of the system and typical trajectory.

Fig. 2 – Schematic phase diagram for $r_s = 0.94$, $r_f = 0.68$ and $\alpha = 0.01$: the region below the line $m_a = \Delta/2$ is not allowed.

The model consists of a particle (or a swarm of noninteracting particles) starting to fall under the action of the gravitational field with an initial velocity (v_x^0, v_y^0), the horizontal and vertical components, respectively, along an inclined line where it bounces inelastically with a restitution coefficient, r, which may vary from 1 if the impact is perfectly elastic to 0 in the totally inelastic case. At each bounce the particle changes its velocity after colliding with a microfacet, which sits on the line and forms an angle $\alpha < 0$ with respect to the horizontal (see fig.1). The collision changes the normal and tangential components of the velocity to the microfacet, $v_{\rm N}$ and $v_{\rm T}$ respectively, according to the rule $v_{\rm N} = -rv'_{\rm N}$ and $v_{\rm T} = rv'_{\rm T}$, where the prime indicates precollisional states.

The transverse and antiparallel components of the velocity with respect to the gravitational acceleration after the n-th impact, v_x^n and v_y^n respectively, can be expressed in terms of the corresponding quantities after the $(n-1)$-th impact by the transformation $\boldsymbol{v}^n = M_n \boldsymbol{v}^{n-1}$:

$$\begin{aligned} v_x^n &= r[\cos^2\alpha - \sin^2\alpha - 4m_n \cos\alpha \sin\alpha] v_x^{n-1} - 2r\cos\alpha \sin\alpha v_y^{n-1} \,, \\ v_y^n &= r[2\cos\alpha\sin\alpha + 2m_n(\cos^2\alpha - \sin^2\alpha)] v_x^{n-1} + r(\cos^2\alpha - \sin^2\alpha) v_y^{n-1} \,, \end{aligned} \tag{1}$$

where m_n is the value of the slope of the inclined plane in the interval between the $(n-1)$-th jump and the successive one. In the case of a constant slope and constant restitution coefficient, one expects the following phenomenology:

a) an accelerated regime, if the restititution coefficient, r, is close to 1 (*i.e.* the system is nearly elastic) and/or the slope is sufficiently large; the particle falls performing larger and larger jumps and accelerates indefinitely;

b) a decelerated regime for smaller values of r and less steeper slopes where the particle eventually comes at rest, since it looses more kinetic energy by collisions than it gains in its fall.

In the plane (m, r), slope-restitution coefficient, the locus representing the motions with vanishing acceleration is a line and thus represents a set of zero measure and only a fine tuning of the parameters may produce a steady velocity state.

In this letter we introduce a twofold modification of the model and assume that:

i) The restitution coefficient is a decreasing function of the impact velocity: namely, $r = r_s$ if the modulus of the impact velocity is smaller than v_c, a threshold value, whereas it is given by $r_f < r_s$ above v_c. Previous studies have shown that the coefficient of restitution, in the presence of elastic and viscoelastic interactions, is not a constant but depends on the impact velocity and has deep repercussions on various aspects of granular dynamics [8]. In the present work for the sake of simplicity we consider a very simple form of the restitution function [9].

ii) The slope of the inclined plane is not constant, but fluctuates about an average value as to reproduce the presence of a rough landscape. We assume that m_n is given by the following law:

$$m_n = m_a + \Delta\left(R_n - \frac{1}{2}\right), \tag{2}$$

where R_n is extracted from a uniform distribution in the unit interval and Δ is chosen so that the slope is nonnegative ($\Delta \leq 2m_a$).

We begin by considering the dynamical "phase diagram" of the model shown in fig. 2. The phase with a constant average velocity is stabilized by the the presence of the two new mechanisms. As shown in fig. 2 in the plane (m_a, Δ) the ball decelerates and comes to rest at low average slopes. On the other hand, the ball accelerates indefinitely for large slopes, but in between it displays a sector where the average velocity along the trajectory is constant. In this region the falling object receives energy from the gravitational field and dissipates some during the collisions with the result of establishing a dynamical equilibrium state in a suitable parameter range. Along the axis $\Delta = 0$ (the fully deterministic system) we can compute exactly, for a given pair of r_s and r_f, the threshold values of the average slope separating, respectively, the stopping phase from the steady velocity phase and the latter from the accelerated phase.

This is achieved by requiring that the largest eigenvalue of the transformation matrix M_n be 1; the result is the formula

$$m_{s(f)} = \frac{[r_{s(f)}(\cos^2\alpha - \sin^2\alpha) - 1]^2 + (2r_{s(f)}\cos\alpha\sin\alpha)^2}{4r_{s(f)}\cos\alpha\sin\alpha}. \tag{3}$$

For a choice of $r_s = 0.94$, $r_f = 0.68$ and $\alpha = 0.01$, we find $m_s = 0.101$ $m_f = 3.775$.

The steady velocity phase represents an attractive fixed point for the dynamics, in the sense that particles with velocities larger than the steady velocity slow down while those with smaller velocities speed up. Small quasiperiodic fluctuations occur about this average value.

It might be worthwhile to mention that the same qualitative behavior has been reported by Ancey *et al.* [2] who compared experiments and theory and studied the dependence on the surface roughness. Their phase diagram showed the existence of two slopes m_1, termed saltational or splashing flow slope, and m_2 related to the friction angle, bounding the steady flow regime in analogy with our findings.

We have focused attention on v_x, the horizontal component of the velocity, because this observable does not change in the interval between two bounces, and studied the behavior of the acceleration with respect to the control parameters r_s, r_f and m_a and Δ. The average acceleration represents a kind of order parameter for the problem. A nonvanishing value of the acceleration characterizes the non–steady-state phase, while a zero value identifies the steady velocity phase. The transition from one regime to the other bears resemblance with the onset of an ordered phase in magnetic model.

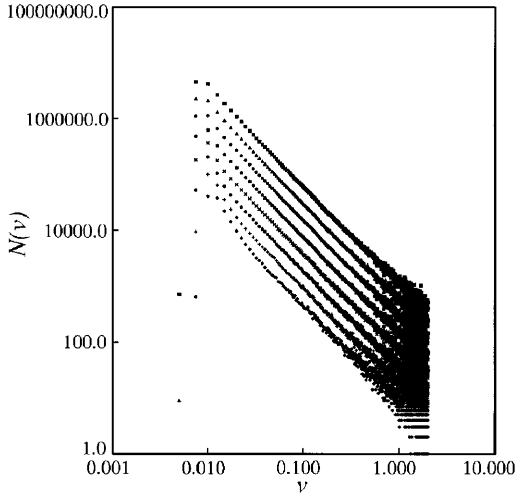

Fig. 3

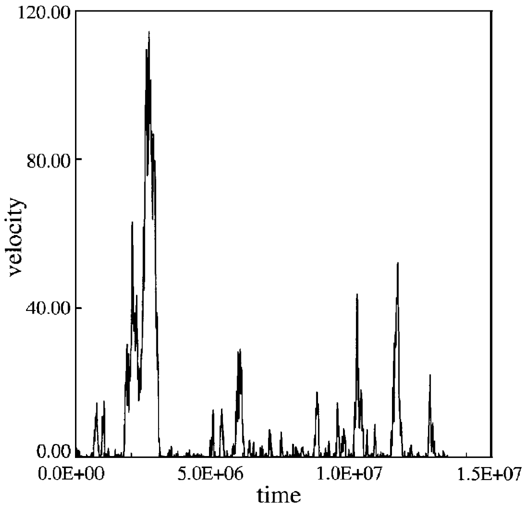

Fig. 4

Fig. 3 – Velocity distribution for a choice of the control parameters $m_a = 3.74$, $\Delta = 7$. $r_s = 0.94$, $r_f = 0.68$ obtained using different coarse-graining sizes.

Fig. 4 – Plot of the horizontal velocity v_x *vs.* time. Notice the presence of fluctuations of all sizes.

In fig. 3 we report the data analysis of the horizontal velocity distribution in the steady phase. The normalized velocity distribution changes as the degree of stochasticity of the surface varies. In order to ascertain the character of the distribution we have performed an analysis of the data by a coarse-graining method, *i.e.* by grouping the velocity after averaging it over time intervals of different length and observed a data collapse. The probability density distribution of the velocity is a power law $P(v_x) \propto v_x^{-\beta-1}$ for large v_x and is characterized by an exponent β which decreases with increasing randomness of the surface. The dependence of β on the slope and roughness is reported in table I. We observe that large velocity fluctuations last longer because the length of the flights increases with velocity and so the energy dissipation per unit time decreases. A similar phenomenon was reported by Taguchi and Takayasu [10] who simulated a bed of powder subject to inelastic collisions and considered the distribution of horizontal velocities. Whereas the distribution appeared to be Maxwellian in the inner layers, in the surface layer the distribution turned out to follow an inverse power law of the velocity.

TABLE I – *Exponents of the velocity distribution for different values of* m *and* Δ.

Δ	$m = 3.71$	$m = 3.73$	$m = 3.75$	$m = 3.77$
5.2	2.125	1.868	1.517	1.102
5.6	2.062	1.817	1.385	1.066
6.0	1.847	1.591	1.360	1.090
6.4	1.765	1.547	1.330	1.069
6.8	1.638	1.460	1.281	1.077
7.2	1.566	1.376	1.278	1.084

Such a power law distribution exists not only at the transition nonaccelerated-accelerated phase, but also within the steady velocity phase, suggesting the presence of self-similar behavior in the fluctuations. Moreover, the frequency of the fluctuations decreases with their size (see fig. 4). All sizes are present, and upon magnifying a given portion of the velocity-time curve one discovers a pronounced self-similarity in this structure which is at the origin of the power law velocity distribution reported in fig. 3. In addition, the duration of the time spent above a given threshold is an increasing function of the maximum velocity reached in the interval. In other words, the distribution of the time intervals during which v_x exceeds the threshold is also a power law of the time interval length.

We consider the axis $\Delta = 0$ where the system is purely deterministic. Moving now away from this axis the system becomes more and more irregular. The velocity distribution broadens as the representative point on the phase diagram moves towards the line $m_a = \Delta/2$, however only for large values of m_a and Δ we clearly observe power laws in the velocity distribution.

In order to understand the origin of the power law behavior, *i.e.* of the self-similarity, we consider the motion generated by the transformation (1). If the velocity is larger than v_c the largest eigenvalue of the 2×2 matrix M_n, calculated for $r = r_f$, turns out to be of the linear form $\lambda \simeq 1 + A(m - m_f)$ for values of m close to m_f, where the positive constant A depends on r. The important thing is that λ, the amplification factor of the velocity, is a random quantity larger or smaller than one with the property that its average logarithm is negative. This follows from the fact that within the steady phase the average value of the slope $m_a < m_f$. In spite of this fact, large values of the velocity can be attained through favorable sequences of slopes with $\lambda > 1$, before the velocity decreases towards zero after a sufficiently large number of bounces as imposed by the condition $\langle \ln \lambda \rangle < 0$. On the other hand, when the velocity decreases below the value v_c the restitution coefficient takes the value r_s and the corresponding eigenvalue of M_n becomes larger than one, thus resetting the velocity to values larger than v_c and avoiding the collapse to zero.

The qualitative behavior of the model can be obtained by exploiting the analogy with the random multiplicative process studied by Levy and Solomon [11, 12] described by the map

$$W_{t+1} = \mu_t W_t \,,$$

where μ_t, playing here the role of the amplification factor λ, is a stochastic variable with a finite support distribution of probability $\Pi(\mu_t)$ and $\langle \ln \mu_t \rangle$ is negative and W_t is constrained to remain larger than a minimum positive value W_0; even in this case, the variable W_t displays a power law distribution, of the same kind of the one we report. For this reason it has been termed convergent multiplicative random process repelled from zero. By introducing an auxiliary variable, x_t, defined by the nonlinear transformation $x_t = \ln W_t$, Solomon and Levy showed that the process can be mapped onto a model of a random walker performing random steps $l_t = \ln \mu_t$ with a drift, $\langle l_t \rangle$, towards a reflecting wall at $-\infty$ and described by the Langevin process:

$$x_{t+1} = x_t + l_t \,.$$

From their analysis it follows that the exponent β depends on the distribution of μ_t and therefore is nonuniversal. In particular if the process converges to a fixed average $W_m = \langle W_t \rangle$ the exponent β is given by

$$\beta = \frac{1}{1 - \dfrac{W_0}{W_m}} \,.$$

This result explains why in our case β varies with the nature of the surface.

To conclude we have studied the statistical properties of an assembly of noninteracting point-like particles subject to the action of a gravitational field and bouncing inelastically on a curve piecewise constant and whose slope has an assigned probability distribution. We have shown that such a system displays already a highly nontrivial behavior even in the absence of any particle-particle interaction and might lead to a better understanding of some dynamical aspects of granular materials [13]. After obtaining the phase diagram of the model in terms of the relevant control parameters, we found that the velocity distribution along the trajectory in the nonaccelerating phase is an inverse power law, characterized by an exponent, β, which decreases as the randomness and the slope increase. What determines the power law behavior is the presence of the nonlinear feedback in the velocity. On the other hand, for small slope the variance of the distribution is very narrow and the distribution delta-like.

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