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VARIATIONAL APPROACH TO THE PHASE DIAGRAM OF RIGID MEMBRANES

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D-dimensional elastic networks randomly embedded in a $d > D$ dimensional euclidean space, are studied employing Hartree (Gaussian) approximation. In presence of an energy depending on the mean curvature this approach leads to the prediction of a phase transition between a flat and a crumpled regime as the bending rigidity decreases in agreement with previous approximate calculations.

1. Introduction

Membranes are two-dimensional extended objects whose properties in the last few years have attracted the attention of condensed matter physicists and field theorists.[1,2] In our view, what renders the physics of such systems appealing to theoreticians is, perhaps, the presence of strong fluctuation effects due to their low dimensionality.[2]

It is necessary, however, to specify that one encounters very different behaviors according to which family of membranes is being considered. In the first family there are the so-called fluid membranes: these have two-dimensional liquid state properties as one can see from the absence of narrow Bragg peaks in their structure factors.[3] The energy of such fluid membranes usually contains a surface area term and a bending elastic contribution, but the constituent molecules can freely flow at their interior and even evaporate.[4]

The second family, on the other hand, is characterized by an infinite strength of the bonds, which join the nodes of a D-dimensional network, and by fixed connectivity; these two properties ensure that solid membranes cannot be torn apart.[5] They can also be seen as a generalization to higher connectivity of the concept of polymer. In the present paper we shall investigate the properties of these crystalline membrane by considering a very simple phenomenological model.

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Let us start by motivating the aims of our approach and stressing the differences with the previous studies on the subject. The general properties of polymerized membranes have been thoroughly studied by means of Monte Carlo simulations,[5] elasticity theory and field theoretical methods.[6,7] The assumption of a continuous elastic medium down to the microscopic scales allows one to apply the machinery of differential geometry and Renormalization Group (RG) methods and to obtain many exact results.[2] This approach, however, requires the knowledge of quantities, such as the surface tension and the elastic constants which are not given by the theory itself, but are extrapolated by continuity of the spatial scale, one of the guiding principles of physics, from the corresponding thermodynamic quantities. To which extent these approaches work on a microscopic scale is often difficult to assess. In particular when first order phase transitions are present there is no *a priori* guarantee that the naive continuum limit catch the essence of the problem since, in that case, there is no diverging characteristic length.

Our model is a simple modification of the so-called tether and bead model,[5] which has the advantage of being both amenable to numerical simulation and to a theoretical although approximate, treatment, without going through an uncontrollable coarse grained description. In other words, we do not assume the existence of an elastic Hamiltonian density, but rather follow an atomistic point of view by introducing a discrete model and by specifying the forces between the particles. Only after averaging over configurations we shall identify the thermodynamic quantities.

Our treatment is based on a self-consistent method, the Hartree approximation, which has already been applied to various problems. We believe that in the present case it can provide insight into the microscopic factors that determine the phase behavior. Of course our approach could be also applied directly assuming an Hamiltonian density since the beginning. A rather similar approach has been recently proposed in Ref. 8.

Let us briefly recall the main results of previous investigations[5-7,9] when an energy curvature term is present favoring locally flat configurations. On the basis of Mean Field and Renormalization Group studies of continuum elastic models of D-dimensional membranes one expects the existence of a phase transition between a low temperature flat phase and a high temperature crumpled phase where the radius of gyration R_G behaves like L^{D/d_F}, with $d_F < D$, as a function of the mass number L^D.[5,7] Whenever d_F, the characteristic fractal dimension of the manifold is different from D, the system is crumpled otherwise it is said to be flat. Although the self-avoidance is the most relevant interaction in the crumpled phase,[4] in the present paper we shall not include it, because we are mainly interested in the flat phase, where, instead, the curvature plays a major role.[3,5] The model we shall discuss presents a high-temperature disordered phase and the latter is characterized, as we shall demonstrate, by a value $d_F = \frac{2D}{2-D}$ for $D < 2$, the typical fractal dimension of membranes without self-repulsion.[4]

It has been pointed out a useful analogy between membranes and the classical Heisenberg magnet, with the normals to the membrane surface playing the role of the local magnetization.[9] However, the resemblance is not complete as indicated both by numerical simulations on a discretized model[5] and by the study of continuous models. The most striking difference is perhaps the result derived by Nelson and Peliti[6] on the basis of an approximate calculation, who found that the effective elastic interaction between the normals to the surface renormalizes the rigidity, thus rendering the flat two-dimensional phase stable, in contrast with the behavior of fluid membranes.[10] The reason is to be found in the long-range nature of the renormalized interaction which makes the hypothesis on which the Mermin and Wagner theorem[11] is based not applicable.

Following further the analogy between membranes and magnets it is natural to explore the limit when the ambient space dimensionality, d, goes to infinity. Within this limit, which has its magnetic counterpart in the well-known spherical model of Berlin and Kac,[12] with the number of components playing the role of d, one finds that when the membrane intrinsic dimension D equals 2 the flat phase becomes unstable. However $1/d$ expansion[7] predicts that a second order crumpling transition occurs at a finite bending rigidity in agreement with the results of Ref. 6. Furthermore the inclusion of excluded volume effects makes the transition first order when $d \leq 219$ according to the Landau theory by Paczuski *et al.*[7]

The Hartree approximation we are going to present allows a non-perturbative study of the crumpling transition for arbitrary d and leads to the prediction that the flat phase exists and it is separated by a first order transition from the high-temperature crumpled region. Unfortunately this self-consistent approximation is unable to predict the conjectured behavior[6] in $d = 3$ of the correlation function for the fluctuations locally normal to the surface. However, it becomes exact in the large d limit and it is interesting in its own right to test the reliability of this widely used variational approach. In this context it is the only method capable to describe the whole phase diagram. Previous analytical approaches were mainly interested in the region near the crumpling transition where the continuum approximation is always employed since the transition is expected to be of second order.[6,7]

The outline of the paper is as follows. Section 2 contains the notation and introduces the model. In Sec. 3 the variational method and the equations for the correlation functions are derived. Section 4 is concerned with the long-wavelength limit of the theory. In Secs. 5 and 6 we discuss the cases of two- and three-dimensional networks in variable number of dimensions of the ambient space. In Sec. 7 we present the conclusions.

2. Definition of Model

Let us consider a system of $N = L^D$ identical particles connected to form a regular D-dimensional hypercubic lattice, $\mathcal{L}$. Each particle, labelled by an internal D component index $i = (i_1, i_2, \ldots, i_D)$, is allowed to move in a d dimensional space

with the only constraint of respecting its connectivity. The model membrane, we shall describe, is composed by all elementary plaquettes of $\mathcal{L}$.

Two body forces act among neighboring particles of the lattice $\mathcal{L}$. By ignoring self-repulsion between distinct elements of the network we write the reduced Hamiltonian as

$$\mathcal{H} = \sum_{(\underline{i},\underline{j})} \left[\alpha(\mathbf{X}_{\underline{i}} - \mathbf{X}_{\underline{j}})^2 + \frac{\lambda}{2}[(\mathbf{X}_{\underline{i}} - \mathbf{X}_{\underline{j}})^2]^2 \right] - \gamma \sum_{\ll \underline{i},\underline{j} \gg} (\mathbf{X}_{\underline{i}} - \mathbf{X}_{\underline{j}})^2 - \frac{\gamma}{2} \sum_{[\underline{i},\underline{j}]} (\mathbf{X}_{\underline{i}} - \mathbf{X}_{\underline{j}})^2 \,. \tag{2.1}$$

The first term describes attractive interactions among particles corresponding to nearest neighbor (n.n) sites in the underlying lattice $\mathcal{L}$. The quartic term, as we shall see below is needed in order to guarantee the overall stability of the system and could be thought as deriving from a Taylor expansion of the interaction energy between nearest neighbor sites. The two negative terms describe interactions between second nearest neighbors have been proposed by Boal and Plischke.[13] The coefficient 1/2 in the fourth term of Eq. (2.1) has been chosen in order to have an isotropic behavior in the continuum limit, as it will be shown in Sec. 4. The idea behind these contributions is to introduce an energy cost which hinders the formation of edges between different regions of the network, thus favoring the presence of nearly flat configurations. In our model they correspond to the bending energy $E_b = -\kappa \sum_{\langle \alpha,\beta \rangle} (\hat{n_\alpha} \hat{n_\beta} - 1)$, originally introduced by Kantor *et al.*,[5] where $\hat{n_\alpha}$ is the normal to the αth triangle of a triangulated surface. Assuming as basis vectors $\underline{b_1} = (1, 0, ., .), \ldots, \underline{b_D} = (0, 0, ., 1)$ we shall denote the sums over next to nearest neighbors (n.n.n) by the symbol $\ll \gg$ if these are of the type

$$\underline{j} = \underline{i} \pm \underline{b_m} \pm \underline{b_n}$$

with m and n integers between 1 and D and $m \neq n$. The symbol [ ] denotes pairs of second neighbors along the axis:

$$\underline{j} = \underline{i} \pm 2\underline{b_1}, \ldots, \underline{j} = \underline{i} \pm 2\underline{b_D} \,.$$

When γ exceeds a critical value say, γ_c, an elementary zero temperature calculation shows the most favorable configuration is the one where the particles form a regular pattern having the same symmetry of the intrinsic lattice $\mathcal{L}$. For the hypercubic lattice let us call ζ the nearest neighbor distance and calculate the total energy per particle:

$$\epsilon_0 = \frac{E_0}{L^D} = D \left[\alpha\zeta^2 - 2\gamma D\zeta^2 + \frac{\lambda}{2}\zeta^4 \right] \,. \tag{2.2}$$

By minimizing the energy with respect to the bond extension ζ one obtains the equilibrium configuration. The first situation corresponds to the crumpled phase, characterized by the equilibrium distance $\zeta = 0$, and occurs when $\gamma \leq \gamma_c = \frac{\alpha}{2D}$. On the other hand, at $\gamma = \gamma_c$ a second order phase transition occurs and when $\gamma \geq \gamma_c$ one finds $\zeta_{sp} = \sqrt{\frac{(2D\gamma - \alpha)}{\lambda}}$. These two regimes are the analogue of the paramagnetic and magnetic phase in the theory of ferromagnets and ζ_{sp} corresponds to the spontaneous magnetization. For values of ζ larger than ζ_{sp} the membrane tends to shrink unless an external tension is applied, whereas for values lower than ζ_{sp} the membrane exerts a force on the boundaries or buckles.

Before turning to a microscopic theory it is important to clarify the meaning of the last two terms in Eq. (2.1) on the basis of simple macroscopic arguments. We consider two configurations of the membrane: the first one is flat, infinitely extended and has an energy per particle given by Eq. (2.2); the second, instead, is an infinitely long open cylinder obtained by bending the network in the direction x, in such a way that every node lies on a circle of radius R, and the distance between nearest neighbors is constant. Let us call ζ the n.n. distance along the axes of the cylinder and d_{nn} and d_{nnn} the n.n. and n.n.n. distances along the planes normal to the axes, respectively and assume these three quantities to be very small with respect to R. The resulting energy per particle is:

$$\epsilon = (D-1)\left[\alpha\zeta^2 - 2\gamma D\zeta^2 + \frac{\lambda}{2}\zeta^4 \right] + \left[(\alpha - 2\gamma(D-1))d_{nn}^2 - \frac{\gamma}{2}d_{nnn}^2 + \frac{\lambda}{2}d_{nn}^4 \right] \,.$$

Expressing d_{nn} and d_{nnn} as functions of the radius and of the angle spanned by the chord joining the sites one obtains:

$$\epsilon = \epsilon_0 + 2R^2 \Big[(\alpha - 2\gamma(D-1))(1 - \cos(\Delta\theta)) - \frac{\gamma}{2}(1 - \cos(2\Delta\theta)) + \lambda R^2 (1 - \cos(\Delta\theta))^2 \Big] \,.$$

Expanding the cosines in powers of $\Delta\theta$ and setting $R\Delta\theta = \zeta = \frac{2\pi R}{L}$ we find the energy excess over the flat phase:

$$\epsilon - \epsilon_0 \approx -\frac{\zeta^2}{12R^2}\left[\alpha\zeta^2 - 2D\gamma\zeta^2 + \lambda\zeta^4 \right] + \frac{\gamma\zeta^4}{2R^2} \,.$$

The right-hand side vanishes when $R \to \infty$ and represents the energy cost due to bending. In the case of vanishing surface tension $\zeta = \zeta_{sp}$ only the second term survives and one can identify $\gamma\zeta^2$ with the curvature elastic constant of the model related to the mean curvature.

From general fluctuation-dissipation arguments one expects that the curvature elastic constants are also important in determining the excitation spectrum in the flat phase. Indeed let us consider fluctuations of the membrane about its flat periodic configuration, in which the nodes are allowed to displace from the equilibrium positions $\bar{X}_{\underline{i}}^{\alpha}$ by an amount $\delta X_{\underline{i}}^{\alpha}$. Upon neglecting anharmonic effects the energy

of a small deformation is

$$\mathcal{H} = \epsilon_0 L^D + \frac{1}{2}\sum_{\underline{i},\underline{j}}\sum_{\alpha,\beta=1}^{d}\left[\frac{\partial^2\mathcal{H}}{\partial X^\alpha_{\underline{i}}\partial X^\beta_{\underline{j}}}\right]_{\mathbf{X}=\bar{\mathbf{X}}}\delta X^\alpha_{\underline{i}}\delta X^\beta_{\underline{j}} .$$

For the particular form of the Hamiltonian we have chosen the matrix of the second derivatives of $\mathcal{H}$ is diagonal in the d-dimensional component sub-space and thus one can easily evaluate the average fluctuation according to the equipartition theorem

$$G^{\alpha,\alpha}(\underline{q}) \equiv \left\langle \delta X^\alpha_{\underline{q}}\delta X^\alpha_{-\underline{q}}\right\rangle = \frac{1}{A^{\alpha,\alpha}(\underline{q})}$$

where $\delta X^\alpha_{\underline{q}} = \frac{1}{L^{D/2}}\sum_j e^{i\underline{q}\underline{j}}$ and

$$\left[\frac{\partial^2\mathcal{H}}{\partial X^\alpha_{\underline{i}}\partial X^\beta_{\underline{j}}}\right]_{\mathbf{X}=\bar{\mathbf{X}}} = \frac{1}{L^D}\sum_{\underline{q}} e^{-i\underline{q}(\underline{i}-\underline{j})}\tilde{A}^{\alpha,\beta}(\underline{q}) .$$

Now, if the perturbation is also slowly varying in space and is a plane wave it is possible to find the explicit form of the long wavelength components of the fluctuation matrix. We shall denote by z the components normal to the hyper-plane of the membrane and by x the parallel components.

$$G^{zz}(\underline{q}) = \frac{k_B T}{\gamma q^4}$$

and

$$G^{xx}(\underline{q}) = \frac{k_B T}{4\lambda\zeta^2 q^2} .$$

One thus sees that the fluctuation of the vertical relative distance between two particles, $(X^z_{\underline{i}} - X^z_{\underline{j}})$, in the long wavelength limit is much larger than the corresponding parallel fluctuation, since to generate area costs no energy.

These simple examples help to illustrate the twofold role of γ, and the asymmetry between the tranverse components and the in-plane components. In fact γ is the 'stiffness' constant which controls the stability of the flat phase and the fluctuations in the directions perpendicular to the plane of the membrane.

However, the above argument assumes the smallness of the fluctuations and its validity becomes questionable when these become very large. In order to take into account the anharmonic character of the fluctuations we shall generalize to the present case a self-consistent method following a procedure similar to that recently introduced by one of us.[14] This approach, while frankly approximate, for finite values of the ambient space dimension d, has the advantage of treating on equal footing the order parameter ζ and its fluctuations and also allows one to calculate the critical value of γ at which the crumpling transition takes place. Finally, we shall demonstrate that within the present approach when $d \to \infty$ we are able to recover the same results of the spherical model recently studied.[15]

3. Variational Method

In order to go beyond perturbation theory we shall consider the variational method for the Hamiltonian (2.1). The approximation scheme, we employ, is characterized by making the free energy itself stationary rather than by expanding it at stationary points of $\mathcal{H}$ as in the saddle point or one-loop approximation. We report only the main steps of the derivation of the constitutive equations and refer the interested reader for the details to Ref. 14. Due to the infinite strength of the bonds no particle reservoir is needed in order to mantain the membrane in chemical equilibrium and a Canonical ensemble description is well-suited. The Boltzmann probability of a given configuration is the $\exp(-\mathcal{H})$, where we have already adsorbed the factor $\beta = \frac{1}{k_B T}$ in the coefficients of the Hamiltonian (2.1). The partition function is the integral over all allowed configurations with the constraint that the centre of mass is fixed. This will eliminate the zeroth mode of the Fourier transform of the $\mathbf{X}_{\underline{i}}$.

We start by introducing the so-called reference Hamiltonian $\mathcal{H}_0$, which depends upon some auxiliary fields $A^{\alpha,\beta}_{i,j}$ and $H^\alpha_{\underline{i}}$, whose role will be clarified later,

$$\mathcal{H}_0 = \frac{1}{2}\sum_{\underline{i},\underline{j}}\sum_{\alpha,\beta=1}^{d} A^{\alpha,\beta}_{i,j}\left(X^\alpha_{\underline{i}} - \bar{X}^\alpha_{\underline{i}}\right)\left(X^\beta_{\underline{j}} - \bar{X}^\beta_{\underline{j}}\right) + \sum_{\underline{i}}\sum_{\alpha=1}^{d} H^\alpha_{\underline{i}}\left(X^\alpha_{\underline{i}} - \bar{X}^\alpha_{\underline{i}}\right) , \quad (3.1)$$

for which one can easily determine the associated reduced free energy, F_0:

$$F_0 = -\ln\left[\int \Pi_{\underline{i},\alpha} dX^\alpha_{\underline{i}} \exp(-\mathcal{H}_0)\right] . \quad (3.2)$$

Explicitly one finds

$$F_0 = -\left(\frac{L^D d}{2}\right)\ln(2\pi) + \frac{1}{2}\ln\det\|A^{\alpha,\beta}_{i,j}\| - \frac{1}{2}\sum_{\underline{i},\underline{j}}\sum_{\alpha,\beta=1}^{d} H^\alpha_{\underline{i}}(A^{-1})^{\alpha,\beta}_{\underline{i},\underline{j}}H^\beta_{\underline{j}} . \quad (3.3)$$

We denote the expectation values with respect the weight $\exp(-\mathcal{H}_0)$ by the symbol $\langle\rangle_0$, and by $\langle\rangle_c$ the connected averages.

The Gibbs inequality implies that $F_1 \equiv F_0 + \langle\mathcal{H} - \mathcal{H}_0\rangle_0$ is never lower than the exact free energy the system described by the Hamiltonian $\mathcal{H}$, for any choice of the trial Hamiltonian $\mathcal{H}_0$. A clear advantage of a quadratic trial Hamiltonian is that expectation values are easily obtainable using factorization rules valid for Gaussian statistical weight.

In order to make the optimal choice, one treats the auxiliary fields $A^{\alpha,\beta}_{i,j}$ as variational parameters by imposing the conditions

$$\frac{\delta F_1}{\delta A^{\alpha,\beta}_{i,j}} = 0 . \quad (3.4)$$

By functionally differentiating with respect to the local field $H_{\underline{i}}^{\alpha}$ the free energy F_0 one obtains the one and two point averages, which are respectively

$$\left\langle X_{\underline{i}}^{\alpha} - \bar{X}_{\underline{i}}^{\alpha} \right\rangle_0 = -\frac{\partial F_0}{\partial H_{\underline{i}}^{\alpha}} = \sum_{\underline{j}} \sum_{\beta} (A^{-1})_{\underline{i},\underline{j}}^{\alpha,\beta} H_{\underline{j}}^{\beta} = 0 \tag{3.5}$$

and the connected two-point average

$$\langle X_{\underline{i}}^{\alpha} X_{\underline{j}}^{\beta} \rangle_c \equiv \left\langle (X_{\underline{i}}^{\alpha} - \bar{X}_{\underline{i}}^{\alpha})(X_{\underline{j}}^{\beta} - \bar{X}_{\underline{j}}^{\beta}) \right\rangle_0 = -\frac{\partial^2 F_0}{\partial H_{\underline{i}}^{\alpha} \partial H_{\underline{j}}^{\beta}} = (A^{-1})_{\underline{i},\underline{j}}^{\alpha,\beta} . \tag{3.6}$$

Upon inverting, we obtain the relation

$$\sum_{\underline{j}} \sum_{\beta} A_{i,j}^{\alpha,\beta} \left\langle X_{\underline{j}}^{\beta} X_{\underline{l}}^{\gamma} \right\rangle_c = \delta_{\underline{i},\underline{l}} \delta^{\alpha,\gamma} . \tag{3.7}$$

Equation (3.7) constitutes together with the explicit solution of Eq. (3.4) the self-consistent field equation which we shall employ in order to determine the phase behavior of the system.

In the presence of first-order phase transitions the evaluation of the total free energy is also necessary and in order to obtain it we substitute Eq. (3.6) in the expression for $F_1 \equiv F_0 + \langle \mathcal{H} - \mathcal{H}_0 \rangle_0$. To summarize we have obtained, in a simple non diagrammatic way, the popular Hartree approximation, often used in the study of field theories with a large number of components.[16] It is well-known that such an approximation becomes exact when the number of components, d, goes to infinity, as we shall see below.

Before concluding this section we remark that the result of the present variational procedure is equivalent to set:

$$A_{i,j}^{\alpha,\beta} = \left\langle \frac{\partial^2 \mathcal{H}}{\partial X_{\underline{i}}^{\alpha} \partial X_{\underline{j}}^{\beta}} \right\rangle_0 \tag{3.8}$$

and is valid for an arbitrary potential. In other words the complicated anharmonic effects are described by means of an effective coupling obtained by averaging self-consistently the force matrix $A_{i,j}^{\alpha,\beta}$. This is much in the same spirit as the Self-Consistent Harmonic Approximation employed in the study of phonons in crystals.[17] By contrast the one-loop approximation to the present problem is recovered if one sets $A_{i,j}^{\alpha,\beta} = \left[\frac{\partial^2 \mathcal{H}}{\partial X_{\underline{i}}^{\alpha} \partial X_{\underline{j}}^{\beta}}\right]_{\mathbf{X}=\bar{\mathbf{X}}}$, or equivalently dropping all the connected averages in Eq. (3.10). In that case we get the last two equations of the previous section.

4. Continuum Limit

In order to obtain an understanding of the phase diagram of the system described in Sec. 2, we first consider the long wavelength properties of the fluctuation spectrum associated with the operator $A_{i,j}^{\alpha,\beta}$. We shall proceed by illustrating the analytic form of the long wavelength soft modes, by Taylor expanding $A_{i,j}^{\alpha,\beta}$ in powers of the lattice constant a and thus solving the coupled self-consistent equations.

As discussed earlier, we characterize the two phases by an order parameter ζ, the crumpled phase corresponding to $\zeta = 0$ whereas the ordered phase to $\zeta > 0$. Due to the symmetry of the lattice one finds

$$\begin{aligned} \left\langle (X_{\underline{i}}^{\alpha} - X_{\underline{i}\pm\underline{b}_1}^{\alpha})(X_{\underline{i}}^{\beta} - X_{\underline{i}\pm\underline{b}_1}^{\beta}) \right\rangle_0 &= \left\langle (X_{\underline{i}}^{\alpha} - X_{\underline{i}\pm\underline{b}_2}^{\alpha})(X_{\underline{i}}^{\beta} - X_{\underline{i}\pm\underline{b}_2}^{\beta}) \right\rangle_0 \\ &= \zeta^2 \delta^{\alpha,\beta} + G_1^{\alpha,\beta} . \end{aligned} \tag{4.1}$$

Using Eq. (3.6) one can calculate $G_1^{\alpha,\beta}$ from $A_{i,j}^{\alpha,\beta}$.

If the nodes of the membrane occupy the sites of a regular hypercubic D-dimensional lattice, that we shall identify with a crystalline phase, then it is convenient to work in momentum space. Let $\tilde{A}$ denote the Fourier transform of A; using the discrete symmetry of the lattice one finds

$$G_1^{\alpha,\beta} = \frac{2}{L^D} \sum_{\underline{q}} (\tilde{A}^{-1})_{\underline{q}}^{\alpha,\beta} (1 - \cos(q_\gamma)) \tag{4.2}$$

independently of $\gamma = 1, \ldots, D$.

In view of the periodic symmetry, we obtain the following expression for the free energy:

$$\begin{aligned} \frac{\Gamma}{L^D} = 2D \left[\frac{\alpha}{2} \zeta^2 - \gamma D \zeta^2 + \frac{\lambda}{4} \zeta^4 \right] - \left(\frac{d}{2} \right) \ln(2\pi) \\ + \frac{1}{2L^D} \sum_{\underline{q}} \left[(d - D) \ln \tilde{A}^T(\underline{q}) + \sum_{\lambda=1}^{D} \ln \tilde{A}^{\lambda,\lambda}(\underline{q}) \right] \\ - \frac{\lambda}{2} D \left[\left(\sum_{\alpha=1}^{d} G_1^{\alpha,\alpha} \right)^2 + 2 \sum_{\alpha,\beta=1}^{d} \left(G_1^{\alpha,\beta} \right)^2 \right] . \end{aligned} \tag{4.3}$$

One then observes $(d - D)$ tranverse degenerate modes or undulations, in which the displacement of the particles is orthogonal to the plane, and D in-plane compression modes or phonons. In practice in the flat phase only two components of the tensor $G_1^{\alpha,\beta}$ are independent, i.e., $G_1^L \equiv G_1^{\lambda,\lambda}$, for $\lambda \leq D$, the in-plane component, and $G_1^T \equiv G_1^{\tau,\tau}$, with $\tau > D$, the transverse component. Performing a Taylor expansion in powers of the lattice constant up to the fourth order one finds

$$\begin{aligned} \tilde{A}^T(\underline{q}) = 2 \left[\alpha - 2D\gamma + \lambda \zeta^2 + \lambda((d - D + 2) G_1^T + D G_1^L) \right] \sum_{\lambda}^{D} \left(q_\lambda^2 - \frac{1}{12} q_\lambda^4 \right) \\ + \gamma \left(\sum_{\lambda}^{D} q_\lambda^2 \right)^2 \end{aligned} \tag{4.4}$$

where q is measured in units of $(\frac{2\pi}{a})$. The corresponding equations for the in-plane components ($\alpha \leq D$) are given by

$$\tilde{A}^{\alpha,\alpha}(\underline{q}) = 2\Big[\alpha - 2D\gamma + \lambda\zeta^2 + \lambda((d-D)G_1^T + (D+2)G_1^L)\Big]\sum_{\lambda}^{D}\left(q_\lambda^2 - \frac{1}{12}q_\lambda^4\right)$$

$$+\gamma\left(\sum_{\lambda}^{D} q_\lambda^2\right)^2 + 4\lambda\zeta^2\left(q_\alpha^2 - \frac{q_\alpha^4}{12}\right) . \qquad (4.5)$$

Notice that $A^T(\underline{q})$ depends only on the square modulus of $\underline{q}$ to second order, whereas the coefficient of the fourth order term is not rotationally invariant.

In the absence of order , $\zeta = 0$, the tensor $\tilde{A}_{\underline{q}}^{\alpha,\beta}$ is diagonal and all the elements are identical. In such a disordered phase a useful indicator of the geometrical structure of the manifold is the gyration radius, R_G, defined as

$$R_G^2 = \frac{1}{2L^{2D}}\sum_{\underline{i},\underline{j}}\left\langle(\vec{X}_{\underline{i}} - \vec{X}_{\underline{j}})^2\right\rangle = \frac{1}{L^D}\sum_{\alpha}\sum_{\underline{q}\neq 0}\frac{1}{(A^{\alpha,\alpha}(\underline{q}))}$$

where we have used the fact that the centre of mass is fixed in order to eliminate the $\underline{q} = 0$ mode. As we shall see below when $\zeta = 0$, $A^{\alpha,\alpha}(\underline{q}) \approx q^2$ in the small q limit and taking into account the infrared cutoff (i.e., $q \geq \frac{2\pi}{L}$) one finds

$$R_G^2 \approx \ln L \; ; \quad D = 2$$

$$R_G^2 \approx L^{2-D} \; .$$

Thus these results are perfectly equivalent to those derived from Gaussian phantom surfaces.[4]

Of special importance is the locus where the surface tension of the system vanishes, because the transverse fluctuations become particularly large. In the flat phase, since the area Ω is proportional to ζ^D, one imposes the condition which identifies this line:

$$\frac{d\Gamma}{d\Omega} = \frac{1}{D\zeta^{D-1}}\frac{d\Gamma}{d\zeta} = 0 \; .$$

It is convenient to work out the derivative using Eq. (4.1) in conjunction with Eqs. (4.3) and (4.4) and keeping $G^{\alpha,\beta}$ fixed at its equilibrium value. Explicitly:

$$\Big[\alpha - 2D\gamma + \lambda(D+2)G_1^L + \lambda(d-D)G_1^T + \lambda\zeta^2\Big]\zeta = 0 \; . \qquad (4.7)$$

One sees from Eq. (4.7) that there exist two types of solutions corresponding to local minima of the free energy: the first, characterized by $\zeta = 0$, describes the crumpled phase. In this case $A^L(\underline{q}) = A^T(\underline{q})$, which signals that transverse fluctuations and phonons become equivalent at the crumpling transition. A mean field type analysis leads to the prediction of a second order phase transition, whereas in the present approximation one obtains a first-order transition for any finite value of d. The disordered phase disappears at the point where the following equation is fullfilled:

$$\Big[\alpha - 2D\gamma_{\rm crit} + \lambda(d+2)G_1\Big] = 0$$

where G_1 denotes the diagonal element of $G_1^{\alpha,\beta}$ in the symmetric phase along one of the directions of the internal lattice $\mathcal{L}$.

The second type of solution of Eq. (4.7) occurs at $\zeta \neq 0$, corresponding to the existence of a flat phase, which is the stable phase well before Eq. (4.8) holds. Thus as γ increases the crumpled phase becomes metastable at a value of $\gamma = \gamma_T$ and a first-order phase transition occurs to the flat phase. This will be discussed in details in the next section for the $D = 2$ case.

5. The Random Surface Model ($D = 2$)

We shall start by studying the most interesting physical application of the model introduced above, namely the two-dimensional membrane and specialize the formulae to the $D = 2$ case . Using Eqs. (4.2) and (4.8) one finds

$$G_1 \approx \frac{1}{2\pi\gamma_{\rm crit}}\ln(L) \; . \qquad (5.1)$$

Finally eliminating G_1 with the help of Eq. (5.1) one obtains the result

$$\gamma_{\rm crit} = \frac{\alpha}{8} + \sqrt{\left(\frac{\alpha}{8}\right)^2 + \frac{\lambda(d+2)}{8\pi}\ln(L)} \; . \qquad (5.2)$$

In other words the value of the rigidity, $\gamma_{\rm crit}$, at which a critical point signals the disappearence of the crumpled phase diverges like the square root of the logarithm of the number of nodes of the membrane. At first sight this situation appears rather disappointing since it seems to indicate the existence of a single disordered phase for all values of γ. This, indeed, was the situation in the case of $d \to \infty$ of Ref. 7. However, as anticipated at the end of the previous section within the present approach, we find that a first-order phase transition takes place before Eq. (5.2) is satisfied, a result which has been often overlooked when discussing the Hartree approximation. Thus, the instability criterion leading to Eq. (5.1) is not

$D = 2$, the point $\zeta = 0$ remains always a local minimum in the free energy, as one can ascertain from the fact that

$$\frac{d^2\Gamma}{d\zeta^2} = \left[\alpha - 2D\gamma + \lambda(d+2)G_1\right]$$

is always positive in the thermodynamic limit and for any finite value of the rigidity parameter γ.

The results found for $D = 2$ and several values of the codimension $(d - D)$ from the numerical solution of the self-consistency Eqs. (4.7) and (4.8) are the following: a disordered phase exists even when the rigidity is large, in agreement with the previous calculation of Ref. 7 for the triangular lattice and $d = \infty$. However, in contrast with their results, because we are considering finite values of d, such a phase is only metastable for values of γ sufficiently large. We obtain, in fact, a stable flat phase for $\gamma > \gamma_T$, where the minimum of $\Gamma(\zeta)$ occurs at $\zeta \neq 0$. At γ_T a first-order phase transition occurs. In Fig. 1 we compare the curve $\frac{\zeta}{\sqrt{d}}$ (in units of $(\frac{\alpha}{\lambda})^{\frac{1}{2}}$ vs. $(\frac{\gamma}{\alpha})$ in the Hartree approximation and in the mean field limit. Notice that in the thermodynamic limit G_1^T remains finite, unlike the mean field case, or the $dm \to \infty$ case, where this quantity diverges as $\ln L$. We can see the above phenomenon as a renormalization of the rigidity $\gamma(q)$, due to the interaction between different q-modes, leading to a $\gamma(q) \approx q^{-\eta}$, in the long-wavelength limit, with $\eta = 2$ in our case.

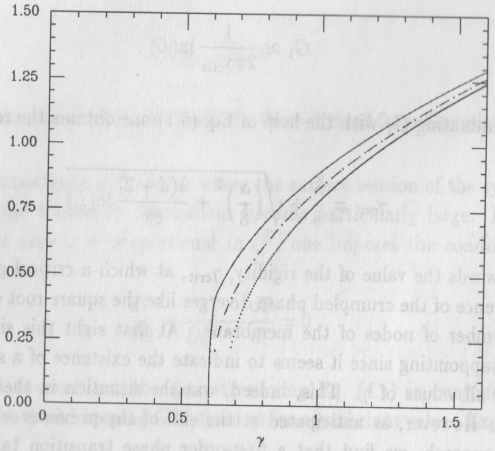

Fig. 1. The scaled order parameter $\frac{\zeta}{\sqrt{d}}$ versus the rigidity parameter γ for $d = 3$ (solid line), for $d = 100$ (dashed line) and $d = 1000$ (dots).

This result should be compared with the Random Phase Approximation treatment of Nelson and Peliti[6] which leads to $\eta = 1$. It is quite interesting that a correct treatment of the Hartree approximation membrane in agreement with the findings of Refs. 5–7.

In the final part of this section we discuss a limit where the model becomes exactly soluble within the Hartree approximation, i.e., the large d limit. This can be done by redefining $\lambda = \frac{\tilde{\lambda}}{d}$ and keeping $\tilde{\lambda}$ finite as $d \to \infty$, in other words, the interaction between each pair of components is very weak. It follows that the present variational method becomes exact when the codimension of the manifold $(d - D)$ goes infinity. In fact, one can show that in the thermodynamic limit the partition function can be evaluated exactly by employing the Hubbard–Stratonovich transformation of the quartic term, followed by a saddle-point estimate of the functional integral. The resulting equations are identical to those obtained via the variational ansatz to the free energy.

This fact is reflected in the long wavelength behavior of the Fourier transform of the transverse correlation function which vanishes exactly as one approaches the line of zero surface tension. Let us consider the limit $d - D \to \infty$ of the equation of state. Within this limit Eq. (4.7) can be rewritten for $D = 2$ as

$$\left[\alpha - 4\gamma + \lambda G_1^T + \frac{\lambda}{d}\zeta^2\right]\zeta = 0 \tag{5.3}$$

where we have neglected the contribution of the longitudinal components. Along the line of zero tension and using Eqs. (4.2) and (4.4) one obtains

$$G_1^T \approx \frac{1}{2\pi\gamma}\ln(L) \tag{5.4}$$

which together with Eq. (5.3) gives

$$\zeta^2 = \frac{(4\gamma - \alpha - \frac{\lambda}{2\pi\gamma}\ln(L))}{\frac{\lambda}{d}} \tag{5.5}$$

thus the flat phase disappears in the limit of $L \to \infty$ for every finite γ, or in other words an infinite bending rigidity is necessary to keep the membrane flat. This effect corresponds to the softening of the bending rigidity due to the long-range interaction between the normals to the surface. As $d \to \infty$ the theory becomes fully consistent in the sense that the correlation functions calculated using the linear response also evident in the small q structure of the function $A^T(q)$. Indeed we have

$$\lim_{q\to 0}\frac{A^T(q)}{q^2} = \frac{1}{DL^D\zeta}\frac{\partial\Gamma}{\partial\zeta}.$$

Some caution must be exercised when applying the Hartree approximation for low values of D and finite d, because it can be often unreliable.

6. The Random Gel Model ($D = 3$)

In the case of a 3-dimensional lattice, two cases are of interest: $d = \infty$ and $d = D = 3$. In both cases we obtain a single correlation function: in the latter $G_1^T = 0$, whereas in the first G_1^L can be neglected. Let us start from $d \to \infty$. By using Eqs. (4.5) and (4.8) in order to locate the critical value of the 'curvature' γ.

$$\gamma_{\rm crit} = \frac{\alpha}{12} + \sqrt{\left(\frac{\alpha}{12}\right)^2 + \frac{\lambda(d+2)}{36\pi^2} q_{\max}} \,. \tag{6.1}$$

Thus the crumpled phase persists down to a finite value of γ, in contrast with the situation found in $D = 2$ and $d > 2$. Notice that, in contrast with the 2-dimensional lattice the relevant fluctuations are the short wave length excitations, thus it would be more appropriate to employ the fully-discretized theory. Following arguments analogous to those used in Sec. 4 one can show that a flat phase, i.e., $\zeta > 0$ does exists even in the limit $d \to \infty$. The equation of state within this limit reads

$$\zeta^2 = \frac{(6\gamma - \alpha - \frac{\lambda}{2\pi^2\gamma} q_{\max})}{\frac{\lambda}{d}} \tag{6.2}$$

and displays a second-order phase transition from the crumpled phase to the ordered phase.

In $d = 3$ case the rigid phase still exists and the phase transition is weakly first order. The equation of state we have studied numerically is

$$\left[\alpha - 6\gamma + 5\lambda G_1^L + \lambda\zeta^2\right]\zeta = 0 \tag{6.3}$$

where G_1^L is given by

$$G_1^{\lambda,\lambda} = \left(\frac{2}{2\pi}\right)^3 \int \frac{d^3q}{A^{\lambda,\lambda}(\underline{q})}(1 - \cos(q_\lambda))$$

with

$$A^{\lambda,\lambda}(\underline{q}) = \left[4\lambda\zeta^2 q_\lambda^2 - \frac{\lambda}{3}\zeta^2 q_\lambda^4\right] + \gamma\left(\sum_\mu^D q_\mu^2\right)^2 . \tag{6.4}$$

7. Conclusions

We have applied the Hartree variational approach to a model of rigid D-dimensional crystalline membrane, embedded in a d-dimensional space, in the absence of excluded volume interactions. This method allows to determine the full phase diagram with the prediction of a crumpling transition[5–8] at a critical value, γ_T, of the rigidity γ. At $\gamma < \gamma_T$ the surface is crumpled with a radius of gyration $R_G \approx L^{D/d_F}$, where $D_F = \frac{2D}{(2-D)}$ is the fractal dimension of the membrane and L^D is the number of nodes. In $D = 2$ $R_G^2 \approx \ln L$. When $\gamma > \gamma_T$ the surface is flat, i.e. $R_G \approx L$. For the $D = 2$ case this findings are in agreement with numerical simulations,[9,10] with $\frac{1}{d}$ expansion,[7] and general arguments.[6] The order of the transition between crumpled and flat phase, predicted by the Hartree approximation is first order at variance with the common belief[5–7] of a continuous transition. This fact does not come as a surprise since it is well-known (see Ref. 14 e.g.) the unreliability of such a method as far as the order of the transition is concerned. It is quite satisfactory, however, the qualitative description of both phases within the same approach, without assuming *a priori* the existence of a flat phase and then justify its stability on the basis of perturbation theory around the flat configuration.[6,7]

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