Eur. Phys. J. Special Topics **179**, 197–206 (2009)
© EDP Sciences, Springer-Verlag 2010
DOI: 10.1140/epjst/e2010-01203-6

THE EUROPEAN PHYSICAL JOURNAL SPECIAL TOPICS

Regular Article

Granular ratchets

G. Costantini[1,2], A. Puglisi[1,2], and U.M.B. Marconi[3,4,a]

[1] CNR (INFM-SMC and ISC), p.le Aldo Moro 2, 00185 Roma, Italy
[2] Università di Roma "La Sapienza", Dipartimento di Fisica, p.le Aldo Moro 2, 00185 Roma, Italy
[3] Università di Camerino, Dipartimento di Fisica, Via Madonna delle Carceri, 62032 Camerino, Italy
[4] INFN, Sez. di Perugia, Italy

Abstract. The general model for granular Brownian ratchets is reviewed. It consists of an asymmetric object moving on a line and surrounded by a two-dimensional granular gas, which is kept in steady state by an external random driving force. In the dilute limit, the two coupled Boltzmann Equations describing the gas and the object are discussed and, with reasonable approximations, a closed system for the main velocity moments is obtained. In the limit of large ratchet mass, a Langevin-like equation for the ratchet velocity is derived, which incorporates an effective force due to dissipation in collision. Fair agreement with simulations under Molecular Chaos assumption is shown.

Introduction

Brownian ratchets are a useful model-system to test our ideas about statistical fluctuations and irreversibility. These devices were originally conceived to rectify thermal fluctuations in non-equilibrium systems with energy conserving dynamics and are able to generate fluctuation-induced currents. The issue, also related to so called Maxwell's demon problem, has been discussed by Smoluchowski and Feynman. A renewed interest for the subject has been motivated by applications of these concepts to molecular biophysics [1], electric and chemical rectifiers [2,3]. Many of these models are mesoscopic and are based on Markovian Langevin dynamics and not on the truly microscopic interaction mechanisms between the ratchet and the surrounding gas. In order to rectify the thermal motion of the particles the microscopic symmetry of detailed balance must be broken, i.e. it necessary to operate under nonequilibrium conditions. To such a purpose, very recently, Van den Broeck and coworkers have studied from a microscopic point of view a Brownian motor and showed that it is possible to obtain rectification of the noise by coupling the ratchet to two thermal reservoirs at different temperatures [4]. They broke two symmetries: time reversal and rotational symmetry, thus obtaining a directed motion. The present authors showed that in the case of a granular gas coupled to a ratchet only *one* reservoir is needed, because inelastic collisions are sufficient to break the microscopic balance. They found that an asymmetric object drifts in a sea of agitated grains and that in the case of a massive tracer the resulting motion can be approximately described by a Langevin equation with a spontaneous drift [5,6]. A not too massive tracer requires a more general description with the introduction of an asymmetric velocity distribution function [7].

In the present paper we first review the problem following the theoretical approach by which we derived the relevant equations of evolution for the tracer and the surrounding gas in the dilute limit. Secondly, under reasonable assumptions, we obtain the general equations for the first relevant velocity moments for the gas and the ratchet. They become uncoupled in the limit of fast gas equilibration: in this case the ratchet dynamics is an unbalanced Markovian process

[a] e-mail: umberto.marinibettolo@unicam.it

which is well described by its first three velocity moments. Finally we discuss the limit of large ratchet mass, where the third velocity moment is negligible and a Langevin equation can be derived. It includes the typical dissipative term proportional to velocity, plus a constant force due to asymmetry and inelastic collisions, and a noise term satisfying a Fluctuation-Dissipation relation with the dissipative term. All predictions are in quantitative agreement with numerical simulations. The model and the coupled gas-ratchet Boltzmann Equations are presented in Section 1, the evolution for the velocity moments are discussed in Section 2, the large ratchet mass limit is discussed in Section 3 and comparison with numerical simulations is shown in Section 4. Finally, conclusions are drawn in Section 5.

1 Theory

We consider a two dimensional rigid convex body of mass M constrained to move along the direction x. It is immersed in a dilute gas of N hard disks of mass m and density $n = N/A$ where A is the area of the box. Its contour, of perimeter C, is characterized by a non-uniform inelasticity, that means that the coefficient of restitution may depend on the point of contact on the surface. The collisions between disks are dissipative with a coefficient of restitution α_{dd}. A thermal bath [8], acting only on the disks, provides the energy necessary to fluidize the system. A collision between the body and a disk changes the pre-collisional velocities $\mathbf{v}' = (v'_x, v'_y)$ and $\mathbf{V}' = (V', 0)$ of the disk and the body, respectively, into their corresponding post-collisional values, $\mathbf{v}$ and $\mathbf{V}$:

$$\begin{aligned} V &= V' + \frac{[1+\alpha(\theta)]\epsilon^2 \sin\theta}{1+\epsilon^2\sin^2\theta}[(v'_x - V')\sin\theta - v'_y\cos\theta] \\ v_x &= v'_x - \frac{[1+\alpha(\theta)]\sin\theta}{1+\epsilon^2\sin^2\theta}[(v'_x - V')\sin\theta - v'_y\cos\theta] \\ v_y &= v'_y + \frac{[1+\alpha(\theta)]\cos\theta}{1+\epsilon^2\sin^2\theta}[(v'_x - V')\sin\theta - v'_y\cos\theta] \end{aligned} \tag{1}$$

where $\epsilon^2 = m/M$ is the mass ratio, $\hat{k}_\parallel$ and $\hat{k}_\perp$ are the unit vectors, parallel and perpendicular respectively, to the contour in the collision point (see Fig. 1). Explicitly $\hat{k}_\parallel = (\cos\theta, \sin\theta)$ and $\hat{k}_\perp = (\sin\theta, -\cos\theta)$ where θ is the angle created by $\hat{k}_\parallel$ and the x axes, modulus 2π (Fig. 1). The coefficient of restitution $\alpha(\theta)$ is a function of θ.

The collision between two disks labeled 1 and 2, instead, is given by

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{v}'_1 - \frac{1+\alpha_{dd}}{2}(\mathbf{v}_{12}\cdot\hat{n})\hat{n} \\ \mathbf{v}_2 &= \mathbf{v}'_2 + \frac{1+\alpha_{dd}}{2}(\mathbf{v}_{12}\cdot\hat{n})\hat{n}, \end{aligned} \tag{2}$$

with $\mathbf{v}_{12} = \mathbf{v}_1 - \mathbf{v}_2$ and $\hat{n}$ the unit vector in the direction joining the centers of the two disks. The stochastic driving force acting on the disks is:

$$m\frac{\partial \mathbf{v}}{\partial t} = -m\Gamma\mathbf{v}(t) + \sqrt{2m\Gamma T_b}\,\zeta(t) \tag{3}$$

with $\zeta(t)$a Gaussian white noise with $\langle\zeta(t)\rangle = 0$ and $\langle\zeta_i(t)\zeta_j(t')\rangle = \delta_{ij}\delta(t-t')$ and $\Gamma \equiv 1/\tau_b$ the drag coefficient [8,10,12]. In the dilute gas limit, where one expects the system to be spatially uniform, we study the evolution of the velocity distribution of the tracer by means of a Boltzmann Equation (BE) for $P(V,t)$ [5]:

$$\frac{\partial P(V,t)}{\partial t} = \int \mathrm{d}V'\,[W(V|V')P(V',t) - W(V'|V)P(V,t)] \tag{4}$$

where the rates for the collision tracer-disk explicitly reads:

$$W(V|V') = \int_0^{2\pi} n\tilde{C}F(\theta)\mathrm{d}\theta \int_{-\infty}^{+\infty} \mathrm{d}v'_x \int_{-\infty}^{+\infty} \mathrm{d}v'_y \phi(v'_x, v'_y, t)\Theta[(\mathbf{V}' - \mathbf{v}') \cdot \hat{k}_\perp](\mathbf{V}' - \mathbf{v}) \cdot \hat{k}_\perp$$
$$\times\delta\left\{V - V' - \frac{\sin\theta}{\kappa(\theta, \alpha(\theta))}\left[(v'_x - V')\sin\theta - v'_y\cos\theta\right]\right\} \tag{5}$$

with $\kappa(\theta, \alpha(\theta)) = (1 + \epsilon^2 \sin^2\theta)/[(1 + \alpha(\theta))\epsilon^2]$, Θ is the Heaviside step function and $\phi(v_x, v_y, t)$ is the pdf of the gas particles. The $F(\theta)$ in Eq. (5) is a shape function of the object and it is such that $dl = \tilde{C}F(\theta)d\theta$ is the length of its outer "effective" contour $\tilde{C}$ that has a tangent between θ and $\theta + d\theta$ (see Fig. 1 for an example).

An analogous equation governs the pdf, $\phi(\mathbf{v}, t)$, of the gas particles obeys, in the dilute limit, a BE analogous to Eq. (4), with the contributions of disk-disk collisions, disk-object collisions and the coupling with the external driving:

$$\frac{\partial\phi(\mathbf{v}, t)}{\partial t} = J[\mathbf{v}|\phi, \phi] + \int \mathrm{d}v'_x \int \mathrm{d}v'_y \, [w(\mathbf{v}|\mathbf{v}')\phi(\mathbf{v}', t) - w(\mathbf{v}'|\mathbf{v})\phi(\mathbf{v}, t)] + \mathcal{B}\phi(\mathbf{v}, t), \tag{6}$$

where $J[\mathbf{v}|\phi, \phi]$ is the Boltzmann collision operator for the disk-disk interactions [15], while $\mathcal{B}$ is an operator representing the effects of the heat bath, explicitly:

$$\mathcal{B}\phi(\mathbf{v}, t) = \Gamma\frac{\partial}{\partial\mathbf{v}}\Big[\mathbf{v}\phi(\mathbf{v}, t)\Big] + Q\,\triangle_v\,[\phi(\mathbf{v}, t)] \tag{7}$$

where $\triangle_v$ is the Laplacian operator with respect to the velocity and $Q = T_b\Gamma/m$.

The transition rate in Eq. (6) is given, instead, by

$$w(\mathbf{v}|\mathbf{v}') = \int_0^{2\pi} \frac{\tilde{C}}{A}F(\theta)\mathrm{d}\theta \int_{-\infty}^{+\infty} \mathrm{d}V P(V', t)(\mathbf{V}' - \mathbf{v}') \cdot \hat{k}_\perp \Theta[(\mathbf{V}' - \mathbf{v}') \cdot \hat{k}_\perp]$$
$$\times\delta\Big\{v_x - v'_x - \frac{\sin\theta}{\epsilon^2\kappa(\theta, \alpha(\theta))}\big[\sin\theta(V' - v'_x) + v'_y\cos\theta\big]\Big\}$$
$$\times\delta\left\{v_y - v'_y + \frac{\cos\theta}{\epsilon^2\kappa(\theta, \alpha(\theta))} \cdot \big[\sin\theta(V' - v'_x) + v'_y\cos\theta\big]\right\} \tag{8}$$

The function $\tilde{C}F(\theta)$ is the same of Eq. (5) because it is connected to the differential cross section that is a property of the colliding couple.

2 Evolution of moments

Instead of solving directly the two BE Eqs. (4) and (6) for the distributions it is convenient to derive a set governing equations for their first few moments using some approximation to close the set. To this purpose we assume a Maxwellian distribution of the disk velocities since the gas is directly coupled to the bath, i.e.

$$\phi(\mathbf{v}, t) = m/(2\pi T_g)\exp[-m(\mathbf{v} - \langle\mathbf{v}\rangle)^2/(2T_g)], \tag{9}$$

where $T_g = m\langle(\mathbf{v} - \langle\mathbf{v}\rangle)^2\rangle/2$ is the granular temperature of the gas. This assumption, which reduces the space of variability of ϕ to the sub-space of Gaussian distributions, leaving only $\mathbf{v}(t)$ and $T_g(t)$ as variables describing the evolution of ϕ, has demonstrated to be realistic in numerical simulations, provided that the surrounding gas is dilute.

Moreover, it has been shown [7] that, if the ratchet mass is comparable or smaller than the disk mass, i.e. $\epsilon^2 \gtrsim 1$, the ratchet pdf is asymmetric and deviates from a Maxwellian distribution, which is characterized by only two parameters, its temperature and average velocity. In order

to capture the asymmetry, we have found that it is essential to consider a further parameter $\xi = \langle (V - \langle V \rangle)^3 \rangle$ [16]. To this aim we assume that the pdf of the tracer can be written as [7]

$$P(V,t) = \sqrt{\frac{M}{2\pi T_r}} \left(1 - \frac{\xi}{6}\frac{\partial^3}{\partial V^3}\right) \exp\left[-\frac{M}{2T_r}(V - \langle V \rangle)^2\right]. \tag{10}$$

After lengthy algebra, and assuming $\langle V \rangle \ll V_{th} \equiv \sqrt{2T_r/M}$, we have derived the following equations for the moments of the tracer

$$\frac{\partial \langle V \rangle}{\partial t} = -\frac{\epsilon^2}{\tau^0_{rd}} \left[\frac{V_{th}}{4} a_1(z) + \frac{a_2(z)}{\sqrt{\pi}} \left(\langle V \rangle - \langle v_x \rangle\right) + \frac{a_3(z)}{\sqrt{\pi}} \langle v_y \rangle + \frac{a_4(z)}{3\sqrt{\pi} V_{th}^2} \xi\right] \tag{11}$$

$$\begin{aligned}\frac{\partial T_r}{\partial t} = \frac{M}{2\tau^0_{rd}} \Bigg[&\frac{V_{th}^2}{\sqrt{\pi}} b_1(z) + \frac{V_{th}}{2} b_2(z) \left(\langle V \rangle - \langle v_x \rangle\right) + \frac{V_{th}}{2} b_3(z) \langle v_y \rangle + b_4 \xi \\ &+ \frac{b_5(z)}{3\sqrt{\pi} V_{th}^2} \xi \left(\langle V \rangle - \langle v_x \rangle\right) + \frac{b_6(z)}{3\sqrt{\pi} V_{th}^2} \langle v_y \rangle \xi\Bigg]\end{aligned} \tag{12}$$

$$\begin{aligned}\frac{\partial \xi}{\partial t} = -\frac{1}{2\tau^0_{rd}} \Bigg[&\frac{3}{4} V_{th}^3 c_1(z) + \frac{V_{th}^2}{2\sqrt{\pi}} c_2(z) \langle V \rangle - \frac{c_3(z)}{2\sqrt{\pi}} \xi - \frac{c_4(z)}{V_{th}} \langle V \rangle \xi \\ &- \frac{V_{th}^2}{2\sqrt{\pi}} c_5(z) \langle v_x \rangle + \frac{V_{th}^2}{2\sqrt{\pi}} c_6(z) \langle v_y \rangle + \frac{c_7}{V_{th}} \langle v_x \rangle \xi - \frac{c_8}{V_{th}} \langle v_y \rangle \xi\Bigg]\end{aligned} \tag{13}$$

and for those of the gas

$$\frac{\partial \langle v_x \rangle}{\partial t} = \frac{1}{N\tau^0_{rd}} \left[\frac{V_{th}}{4} a_1(z) + \frac{a_2(z)}{\sqrt{\pi}} \left(\langle V \rangle - \langle v_x \rangle\right) + \frac{a_3(z)}{\sqrt{\pi}} \langle v_y \rangle + \frac{a_4(z)}{3\sqrt{\pi} V_{th}^2} \xi\right] - \Gamma \langle v_x \rangle \tag{14}$$

$$\frac{\partial \langle v_y \rangle}{\partial t} = -\frac{1}{N\tau^0_{rd}} \left[\frac{V_{th}}{4} d_1(z) + \frac{a_3(z)}{\sqrt{\pi}} \left(\langle V \rangle - \langle v_x \rangle\right) + \frac{d_2(z)}{\sqrt{\pi}} \langle v_y \rangle + \frac{d_3(z)}{3\sqrt{\pi} V_{th}^2} \xi\right] - \Gamma \langle v_y \rangle \tag{15}$$

$$\begin{aligned}\frac{\partial T_g}{\partial t} = \frac{m}{2N\tau^0_{rd}} \Bigg[&\frac{V_{th}^2}{2\sqrt{\pi}} e_1(z) + \frac{V_{th} e_2(z)}{4} \left(\langle V \rangle - \langle v_x \rangle\right) + \frac{V_{th}}{4} e_3(z) \langle v_y \rangle + \frac{e_4}{2} \xi \\ &+ \frac{e_5(z)}{3\sqrt{\pi} V_{th}^2} \left(\langle V \rangle - \langle v_x \rangle\right) \xi + \frac{e_6(z)}{3\sqrt{\pi} V_{th}^2} \langle v_y \rangle \xi\Bigg] - \frac{v_{th}^2}{2\tau_{dd}} (1 - \alpha_{dd}^2) - 2\Gamma v_{th}^2 + 4Q\end{aligned} \tag{16}$$

where $\tau^0_{rd} = (n\tilde{C} V_{th})^{-1}$ is an estimation of the ratchet-disk collision time, $v_{th} = \sqrt{2T_g/m}$ is the thermal velocity of the disks, $\tau_{dd} = (\sqrt{2\pi} n \sigma v_{th})^{-1}$ is the collision time between two disks. The coefficients of the above equations have been given elsewhere [17]. Comparing the Eqs. (11) and (14) we obtain that

$$\frac{\partial \langle v_x \rangle}{\partial t} = -\frac{1}{N\epsilon^2} \frac{\partial \langle V \rangle}{\partial t} - \Gamma \langle v_x \rangle. \tag{17}$$

In the steady state the gas average velocity component $\langle v_x \rangle = 0$ vanishes. Due to the tracer symmetry about its horizontal axis also the gas vertical velocity component, $\langle v_y \rangle = 0$, vanishes. In this situation, the average velocity of the object, $\langle V \rangle$ is given by

$$\langle V \rangle = -V_{th} \left[\frac{\sqrt{\pi}}{4} \frac{a_1(z)}{a_2(z)} + 3 \frac{\xi}{V_{th}^3} \frac{a_4(z)}{a_2(z)}\right]. \tag{18}$$

The contribution of ξ is important if $\epsilon^2 \gtrsim 1$, but negligible in the opposite limit. In this case the average velocity is determined by the ratio $a_1(z)/a_2(z)$.

In conclusion, the asymmetry of the system, appearing in the coefficients a_1, a_2, and a_4, determines a net drift of the object: a study of these coefficients shows that a modulation of the inelasticity along the surface, i.e. a non-constant $\alpha(\theta)$, is more efficient in producing a net drift, with respect to the geometrical asymmetry. So far the theory is general, since we did not make any assumption about the mass ratio ϵ^2. However, the results strongly depend on the limit we consider.

2.1 Time scales

It is interesting, looking at Eqs. (11)–(16), to discuss the degree of coupling between the gas, the object and the thermal bath, which is determined by the three characteristic times present in the system: the relaxation time of the thermal bath τ_b, the disk-disk collision time τ_{dd} and the disk-ratchet collision time τ_{dr}. Here, we are interested in the case $\tau_{dd} < \tau_b$, where inelastic collisions among the gas particles become relevant; in this case, three scenarios can occur: (i) when $\tau_{dr} < \tau_{dd} < \tau_b$, (ii) when $\tau_{dd} < \tau_{dr} < \tau_b$ or (iii) when $\tau_{dd} < \tau_b < \tau_{dr}$. In case (i), the dynamics of the ratchet and the disks are strongly coupled: in this case we expect the region of gas surrounding the object to be correlated with the object itself, making doubtful the assumptions of homogeneity and diluteness introduced at the beginning to treat the system with Molecular Chaos. If, instead, the disk-disk collisions are more frequent, i.e. in cases (ii) and (iii), fast and homogeneous relaxation of the gas is expected. In particular, in situation (iii) the gas dynamics is dominated by internal collisions and by the external driving, and can be regarded as uncoupled from the ratchet: in this limit the gas velocity pdf $\phi(\mathbf{v})$ is constant and Eq. (4) is a Master Equation, i.e. the ratchet velocity performs a Markov process where detailed balance, i.e.

$$P(V')W(V|V') = P(-V)W(-V'|-V) \tag{19}$$

is not satisfied [18], unless $\alpha(\theta) \equiv 1$. Note that the condition for the occurrence of case (i) can be estimated by

$$\frac{\tau_{dd}}{\tau_{dr}^0} = \frac{1}{N}\sqrt{\frac{\pi\eta}{2}}\sqrt{\frac{\rho_d}{\rho_r}} \gg 1 \tag{20}$$

where $\tau_{dr}^0 = N\tau_{rd}^0$, $\rho_d = 4m/(\pi\sigma^2)$ and $\rho_r = 4\pi M/\tilde{C}^2$ are proportional to the mass densities of the disks and the ratchet, respectively. Then case (i) occurs when $\rho_d/\rho_r > N^2$ and this never takes place, in practice. Cases (ii) and (iii), instead, are characterized by the ratio τ_{dr}^0/τ_b being smaller or larger than 1, respectively. Because $\tau_{dr}^0 \propto A/\tilde{C}$ and $A \gg \tilde{C}$, in order to neglect the excluded area of the object, the case (ii) is obtained only if τ_b is large enough.

3 Limit of massive tracer

In the case of a large mass of the tracer, numerical simulations indicate that the tracer velocity distribution is always close to a Gaussian: for this reason one is induced to consider the first two terms of the Kramer-Moyal expansion

$$\frac{\partial P(V,t)}{\partial t} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}V}\right)^n [j_n(V)P(V,t)] \tag{21}$$

where $j_n(V) = \int \mathrm{d}V'(V'-V)^n W(V'|V)$ [19]. By retaining only the first two terms we obtain the sought Fokker-Planck Equation, which can be still simplified by expanding these terms in the small parameter ϵ. If the tracer is symmetric with respect to the x axes, a closed Fokker-Planck equation is obtained, corresponding to the following Langevin equation:

$$\dot{V}(t) = -\gamma V(t) + \frac{\mathcal{F}}{M} + \Gamma(t) \tag{22}$$

with noise $\langle \Gamma(t)\Gamma(t')\rangle = \frac{2\gamma T_r}{M}\delta(t-t')$, $\langle \Gamma(t)\rangle = 0$ and effective parameters

$$\gamma = \epsilon n\tilde{C}\sqrt{\frac{2T_g}{\pi M}}B_2 \tag{23}$$

$$\frac{\mathcal{F}}{M} = -\frac{n\tilde{C}T_g}{2M}\Big[B_1 + \epsilon^2(\eta-1)B_3\Big]. \tag{24}$$

The coefficients B_n are defined as

$$B_n \equiv \int_0^{2\pi} \mathrm{d}\theta F(\theta) \sin^n \theta [1 + \alpha(\theta)]. \tag{25}$$

The Eq. (24) shows that, in the case of a massive tracer, the predominant contribution to the force is given by B_1 that originates from a non-uniform inelasticity $\alpha(\theta)$. In the case, in fact, of a constant coefficient of restitution, B_1 vanishes and the force results weak being proportional to ϵ^2. The drift results determined in this situation by a mix of two ingredients: the spatial asymmetry and the different temperature between the tracer and the granular gas. We remark that while the average velocity $\langle V \rangle$ does not depend explicitly on the perimeter length of the ratchet, the effective friction coefficient, γ and effective force F do depend. Explicit formulae can be obtained for an isosceles triangular ratchet with angle $2\theta_0$ and with $\alpha_1 = \alpha_2 = \alpha$, we have for $\epsilon^2 \ll 1$

$$\langle V \rangle = -\frac{1-\eta}{4}\sqrt{\frac{2\pi T_g}{m}}\epsilon^2(1 - \sin\theta_0). \tag{26}$$

and for a flat "piston" with the two faces with different inelasticities α_1, for the left face, and α_2, for the right face, one retrieves in the same limit [7]:

$$\langle V \rangle = -\sqrt{\frac{2\pi T_g}{m}}\frac{\alpha_2 - \alpha_1}{4(2 + \alpha_2 + \alpha_1)}\left[1 + \epsilon^2(\eta - 1)\right] \tag{27}$$

which, in general, has a larger signal-noise ratio $\langle V \rangle / \sqrt{T_g}$, with respect to the uniformly inelastic case (26), and should be easier to be observed in experiments.

4 Numerical simulations

In the present section we compare the previous theoretical predictions with numerical results: the two basic assumptions, i.e. diluteness (which implies Molecular Chaos) and Markovianity for the tracer are directly enforced in our algorithm. We are left with a few other non-trivial assumptions, which are checked in this section by comparing theory and numerics: in particular we see that

- the ratchet transition rates $W(V|V')$ are in good agreement with formula derived from the assumption of Gaussian gas particle velocities;
- the third moment of the velocity of the ratchet is sufficient (and necessary) to describe its steady state for a generic value of its mass;
- in the limit of massive ratchet, beyond the agreement with drift and ratchet velocity variance, we also find that the velocity autocorrelation is in good agreement with the effective Langevin equation, confirming that fluctuations are well described by this equation and the Fokker-Planck truncation of the Kramers-Moyal expansion is reasonable.

All these numerical results are obtained, for the sake of simplicity, considering as object an equilateral triangle with different inelasticities for the left face and for the two right faces (see Fig. 1). In particular, we assume the coefficient of restitution to be α_2 if $0 \leq \theta \leq \pi$, and α_1 if $\pi \leq \theta \leq 2\pi$. The collision rules (1) are well defined if the surface is smooth. We consider then a triangle with the vertex shaped by circular arches of radius $\sigma_v/2$. In this case the outer surface is $\tilde{C} = 3L + \pi(\sigma + \sigma_v)$ where L is the "linear" side of the triangle (see Fig. 1). The function $F(\theta)$, using the symmetry of the object, becomes

$$F(\theta) = \frac{L}{\tilde{C}}\left[\delta\left(\theta - \frac{3}{2}\pi\right) + 2\delta\left(\theta - \frac{\pi}{6}\right)\right] + \frac{r_e}{\tilde{C}} \tag{28}$$

with $r_e = (\sigma + \sigma_v)/2$.

Moreover, the transition rate (5) can be written as

$$W(V|V') = n\tilde{C}\sqrt{\frac{m}{2\pi T_g}}\left[(V' - V)\Theta(V' - V)g_-(V, V') + (V - V')\Theta(V - V')g_+(V, V')\right] \tag{29}$$

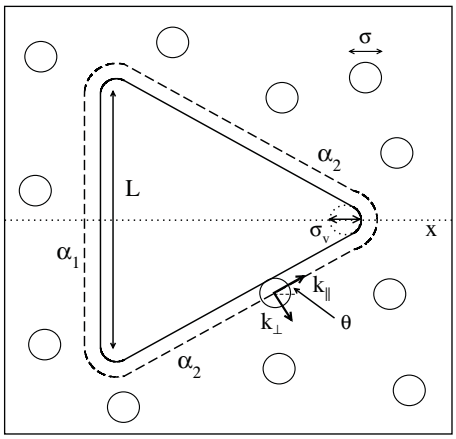

Fig. 1. Sketch of an inelastic equilateral triangle with different coefficient of restitution on left and right side (α_1 and α_2 respectively) and surrounding by a gas of inelastic particles. The dashed line correspond to the effective triangle seen by a colliding disk.

where

$$g_-(V,V') = \int_{\sin\theta>0} \mathrm{d}\theta F(\theta) \frac{\kappa^2(\theta,\alpha_2)}{\sin^2\theta} \exp\left\{-\frac{m\sin^2\theta}{2T_g}\left[\frac{\kappa(\theta,\alpha_2)}{\sin^2\theta}(V-V')+V'\right]^2\right\} \equiv$$
$$\equiv \int_{\sin\theta>0} \mathrm{d}\theta F(\theta)\ \lambda(\theta,\alpha_2,V,V') \tag{30}$$

$$g_+(V,V') = \int_{\sin\theta<0} \mathrm{d}\theta F(\theta)\ \lambda(\theta,\alpha_1,V,V'). \tag{31}$$

The last line of Eq. (30) defines the function $\lambda(\theta,\alpha(\theta),V,V')$. Using the Eq. (28) and the symmetry with respect to θ, the above expressions become

$$g_-(V,V') = \frac{2L}{\tilde{C}}\lambda\left(\frac{\pi}{6},\alpha_2,V,V'\right) + \frac{r_e}{\tilde{C}}\int_0^\pi \mathrm{d}\theta\ \lambda(\theta,\alpha_2,V,V') \tag{32}$$

$$g_+(V,V') = \frac{L}{\tilde{C}}\lambda\left(\frac{\pi}{2},\alpha_1,V,V'\right) + \frac{r_e}{\tilde{C}}\int_0^\pi \mathrm{d}\theta\ \lambda(\theta,\alpha_1,V,V'). \tag{33}$$

Some sections of the transition rate surface, for different values of ϵ^2, are shown in Fig. 2, together with the simulation data obtained from a DSMC [20] with $\tau_b/\tau_{dd} = 137.5$, $\alpha_1 = 1.0$ and $\alpha_{dd} = 0.9$. The Figure 2 displays a very good agreement between theory and DSMC for the transition rate for all values of α_2 and ϵ^2 studied.

In Fig. 3 we show the rescaled observable $\langle V\rangle/V_{th}$ as a function of the right coefficient of restitution α_2 for different values of ϵ^2. The theory results in good agreement with the simulation data, supporting our assumptions. The trends are analogous to those obtained in [7], where only the inelasticity asymmetry was considered: such a similarity indicates that the effects, due to the different inelasticity of the ratchet, are predominant with respect to the geometrical asymmetry, that however can be not neglected [5]. The data referring to ratio η of the granular temperatures of the system go in the same direction (see Fig. 4).

Finally in Fig. 5 we have displayed the tracer velocity self-correlation

$$C(t) = \langle (V(t) - \langle V\rangle)(V(0) - \langle V\rangle)\rangle, \tag{34}$$

for a massive equilateral triangle with $\epsilon^2 = 0.01$ together with the theoretical prediction $T_r/M\exp(-\gamma t)$. It gives a direct measure of γ and agrees well with the theory for $\epsilon^2 \ll 1$.

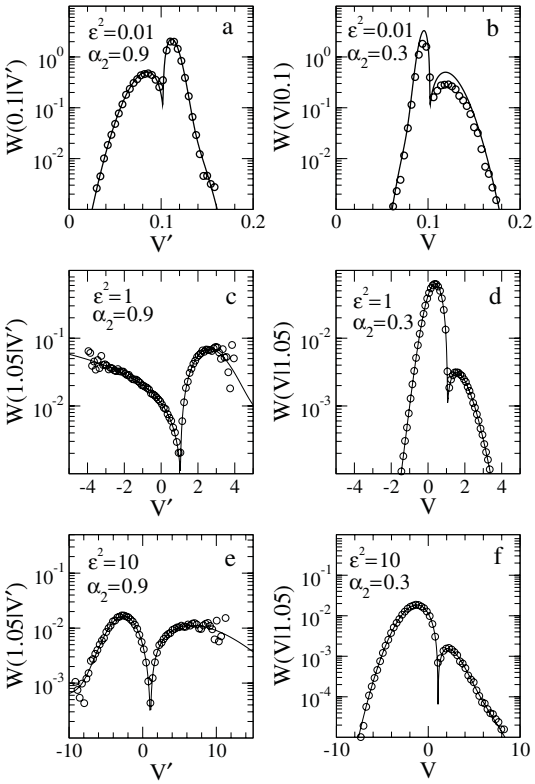

Fig. 2. The transition rate $W(V|V')$ of an inelastic equilateral triangle for different values of the mass ratio ϵ^2 and the coefficient of restitution α_2. The panels a, c and e show the trend as a function of V' for $\alpha_2 = 0.9$, $\alpha_1 = 1.0$, $\alpha_{dd} = 0.9$ and $\epsilon^2 = 0.01, 1$ and 10 respectively. The panels b, d and f instead show the trend as a function of V for $\alpha_2 = 0.3$ and $\alpha_1 = 1.0$ and for the same values of ϵ^2. The symbols correspond to the simulation data, while the lines are obtained from Eqs. (29), (32) and 33).

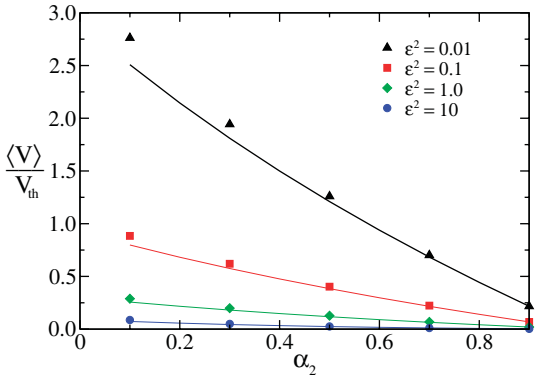

Fig. 3. The velocity of an equilateral triangle ratchet, rescaled with its thermal velocity $V_{th} = \sqrt{T_r/M}$, as function of the coefficient of restitution α_2 for $\alpha_{dd} = 0.9$, $\alpha_1 = 1.0$, $\tau_b/\tau_{dd} = 137.5$ and for different values of the parameter ϵ^2: 10 (circles), 1.0 (diamonds), 0.1 (squares) and 0.01 (triangles). The symbols correspond to the simulation data while the lines correspond to the solutions obtained from Eqs. (11)–(16).

For large inelasticities, at large time, $C(t)$ shows a slightly fatter tail, a feature observed in previous studies of granular gases [22]. The exponential decay of $C(t)$ also implies diffusive behavior at large times. In particular, from Eq. (22) one can predict a diffusion coefficient

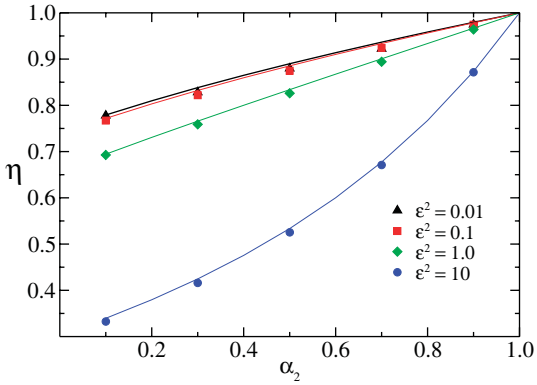

Fig. 4. The temperature ratio $\eta = T_r/T_g$ as function of the coefficient of restitution α_2 for the same cases of Fig. 3. The symbols correspond to the simulation data while the lines correspond to the solutions obtained from Eqs. (11)–(16).

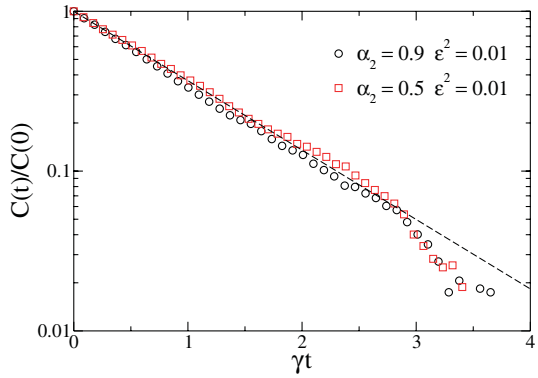

Fig. 5. Rescaled self-correlation functions of the tracer velocity against rescaled time γt, with $\epsilon^2 = 0.01$, $\alpha_1 = 1.0$ and two different choices of inelasticity: $\alpha_2 = 0.9$ (circles) and $\alpha_2 = 0.5$ (squares). The bold dashed line is the theoretical prediction coming from Eq. (22).

$D = T_r/(M\gamma)$ and a general formula for the mean squared displacement $\mathcal{D}_2 = \langle (d(t) - \langle d(t) \rangle)^2 \rangle$, with $d(t) = X(t) - X(0)$, which reads: $\frac{\mathcal{D}_2(t)\gamma}{2D} = z - [1 - \exp(-z)]$ where $z = \gamma t$. This prediction is fairly confirmed by MD and DSMC simulations, see [5].

5 Summary and conclusions

In this paper we have reviewed the problem of rectifying the random motion of a massive object moving in a granular fluid, the so called granular ratchet. Key ingredients to construct such a device is an asymmetry in their geometrical shape or a non uniformity in their inelasticity profile breaking the detailed balance condition. The resulting directed motion is assimilable in the limit of heavy ratchets to a Langevin process with drift. However, in general, the system must be described by a coupled pair of Boltzmann-like equations for the velocity distributions of the ratchet and of the fluid. We have obtained an approximate solution of such equations by considering the evolution of the first few moments of these distributions. In particular, we have found that in the limit of light ratchets it is fundamental to include the third moment of the velocity pdf of the ratchet because this is not Gaussian. The theoretical predictions have been compared with DSMC simulations and a good agreement found. Besides studying average quantities we have also considered finer properties such as the transition rates and verifies that the theory makes excellent predictions even in this case. Finally a ratchet behavior is expected

to be observed in experiments; simplest geometries (a disk of two materials, or a rotating wheel) can be used, low signal/noise ratio is the only problem to be practically solved.

The work of GC and AP is supported by the "Granular-Chaos" project, funded by the Italian MIUR under the FIRB-IDEAS grant number RBID08Z9JE.

References

1. C.S. Peskin, G.M. Odell, G.F. Oster, Biophys. J. **65**, 316 (1993)
2. R.D. Astumiam, M. Bier, Phys. Rev. Lett. **72**, 1766 (1994)
3. J. Prost, J.-F.M. Chauwin, L. Peliti, A. Ajdari, Phys. Rev. Lett. **72**, 2652 (1994)
4. C. Van den Broeck, R. Kawai, P. Meurs, Phys. Rev. Lett. **93**, 090601 (2004)
5. G. Costantini, U.M.B. Marconi, A. Puglisi, Phys. Rev. E **75**, 061124 (2007)
6. B. Cleuren, C. van den Broek, Europhys. Lett. **77**, 50003 (2007)
7. G. Costantini, U.M.B. Marconi, A. Puglisi, Europhys. Lett. **82**, 50008 (2008)
8. A. Puglisi, V. Loreto, U.M.B. Marconi, A. Petri, A. Vulpiani, Phys. Rev. Lett. **81**, 3848 (1998)
9. A. Puglisi, V. Loreto, U.M.B. Marconi, A. Vulpiani, Phys. Rev. E **59**, 5582 (1999)
10. F. Cecconi, F. Diotallevi, U.M.B. Marconi, A. Puglisi, J. Chem. Phys. **120**, 35 (2004)
11. F. Cecconi, U.M.B. Marconi, F. Diotallevi, A. Puglisi, J. Chem. Phys. **121**, 5125 (2004)
12. D.R.M. Williams, F.C. MacKintosh, Phys. Rev. E **54**, R9 (1996)
13. T.C.P. van Noije, M.H. Ernst, Granular Matter **1**, 57 (1998)
14. I. Pagonabarraga, E. Trizac, T.C.P. van Noije, M.H. Ernst, Phys. Rev. E **65**, 011303 (2001)
15. V. Garzó, J.M. Montanero, Physica A **313**, 336 (2002)
16. N. Sela, I. Goldhirsch, Phys. Fluids **7**, 507 (1995)
17. G. Costantini, A. Puglisi, U.M.B. Marconi, J. Stat. Mech., P07004 (2009)
18. A. Puglisi, P. Visco, E. Trizac, F. van Wijland, Phys. Rev. E **73**, 021301 (2006)
19. H. Risken, *The Fokker-Planck Equation. Methods of Solution and Applications*, Vol. 18 (Springer Series, Springer-Verlag, Berlin-Heidelberg, New York, Tokyo, 1984)
20. G.A. Bird, *Molecular Gas Dynamics and the Direct Simulation of Gas Flows*, edited by J.M. Montanero, A. Santos (Clarendon Oxford, 1994)
21. G.A. Bird, Granular Matter **2**, 53 (2000)
22. T.C.P. van Noije, M.H. Ernst, R. Brito, Physica A **251**, 26683 (1998)