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GROWTH KINETICS IN THE Φ^6 N-COMPONENT MODEL. NONCONSERVED ORDER PARAMETER

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The dynamics of a system with a nonconserved N-component order parameter described by a Ginzburg–Landau free energy functional containing a sixth order nonlinearity is discussed in the spherical limit. The model displays a richer behavior than the well-studied Φ^4 case. In particular it shows interesting properties regarding metastability, a feature which is absent in the spherical Φ^4 model.

1. Introduction

In recent years there has been a considerable progress in understanding the dynamics of the approach to equilibrium of physical systems subject to a sudden quench from a high temperature condition to a final subcritical state. After the quench domains of the equilibrium phases spontaneously form and their typical size $L(t)$ increases in time.

A unifying feature of growth kinetics phenomena, which has emerged from experiments and numerical and theoretical studies, is the existence of the so-called dynamical scaling in the late stage of phase separation. One observes that a single dominant length $L(t)$ is asymptotically present in the system and all the physical quantities depend on time through $L(t)$ only. According to the standard scaling hypothesis[3] the structure factor obeys the asymptotic form

$$C(\mathbf{k},t) = L^{\alpha}(t)F(\mathbf{k})L(t)\,, \qquad (1)$$

where $\alpha = d$, the spatial dimensionality, for quenches inside the ordering region. It has been shown,[1] however, that for an N-component conserved order parameter, the scaling symmetry is lost in the large N limit and is replaced by a more complex multiscaling behavior.

The present paper is concerned with the dynamics of an N-component vector spin model, described by an $O(N)$ invariant Ginzburg–Landau free energy functional containing nonlinear terms up to the sixth order in the field. The model is exactly solved in the large N limit, in the case of a nonconserved order parameter (NCOP or model A), at zero temperature. The more lengthy case of a conserved order parameter (COP or model B) is discussed in a subsequent paper.

Whenever the local potentials of the Φ^4 and Φ^6 models display similar landscapes, i.e. an equal number of minima (see Fig. 1), and stable equilibrium is attained, the asymptotic behavior is the same. In particular, in the ordering region, standard scaling is always obeyed with identical exponents and scaling functions.

On the other hand, the present model shows novel features regarding metastability. It is commonly believed that the metastable solutions are lost when taking the limit $N \to \infty$. This behavior is expected because the linearization procedure of the spherical model transforms the potential into a new one with a single well. In this paper we show that this is not the case and that metastable relaxation is possible, under appropriate conditions, even in the large N limit, due to the time dependence of the effective potential. Such a feature is not shared by the Φ^4 model. Therefore, at $T = 0$, since thermal fluctuations are frozen and a final metastable state is indefinitely maintained in time, the two models can be very different as regards their asymptotic behavior.

This paper is organized into four sections. In Sec. 2 we set the notation, introduce the model, and deduce the dynamical equations obeyed by the order parameter field in the large N limit. These are exactly solved in Sec. 3, for the NCOP case, with zero temperature. We then discuss briefly the results and present the conclusions in the fourth part.

2. The Model

We consider a system described by an N-dimensional vector field $\mathbf{\Phi}(\mathbf{r},t) = [\Phi_1(\mathbf{r},t), \ldots, \Phi_N(\mathbf{r},t)]$ prepared in a state $P[\Phi, T_i]$ at the initial temperature T_i. At the time $t = 0$ this system is suddenly quenched to a lower temperature $T_f < T_i$ and relaxation to equilibrium proceeds via the Langevin equation of motion

$$\frac{\partial \Phi_\alpha(\mathbf{r},t)}{\partial t} = -\Gamma(\mathbf{r}) \frac{\partial \mathcal{H}[\Phi,\mu]}{\partial \Phi_\alpha(\mathbf{r},t)} + \eta_\alpha(\mathbf{r},t)\,, \qquad (2)$$

where μ is a set of parameters specifying the free energy functional, the internal index α varies from 1 to N, and the kinetic coefficient $\Gamma(\mathbf{r})$ takes a constant value Γ for NCOP while it is given by $-\Gamma\nabla^2_{\mathbf{r}}$ for COP.

The term $\boldsymbol{\eta}(\mathbf{r},t)$ is a Gaussian noise having zero average and autocorrelation function

$$\langle \eta_\alpha(\mathbf{r},t)\eta_\beta(\mathbf{r},t)\rangle = 2\Gamma(\mathbf{r})T_f\delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')\,. \qquad (3)$$

In the following we shall consider the model described by a free energy functional of the form

$$\mathcal{H}[\Phi] = \int d^d r' \left[\frac{1}{2}(\nabla\Phi)^2 + \frac{r}{2}(\Phi^2) + \frac{g}{4N}(\Phi^2)^2 + \frac{\lambda}{6N^2}(\Phi^2)^3\right]. \qquad (4)$$

According to Eq. (2) the motion of the generic field component α is described by

$$\frac{\partial \Phi_\alpha}{\partial t} = -\Gamma(\mathbf{r})\left[-\nabla^2 + r + \frac{g}{N}\sum_{\beta=1}^{N}\Phi_\beta^2 + \lambda\left(\frac{1}{N}\sum_{\beta=1}^{N}\Phi_\beta^2\right)^2\right]\Phi_\alpha + \eta_\alpha(\mathbf{r},t) \qquad (5)$$

with $\mu = (r, g, \lambda)$. The dynamics of the field Φ is thus determined by the potential (see Fig. 1)

$$V(|\Phi|) = \frac{r}{2}\Phi^2 + \frac{g}{4N}(\Phi^2)^2 + \frac{\lambda}{6N^2}(\Phi^2)^3 \qquad (6)$$

through the set of parameters μ entering Eq. (5).

As it is well known,[2] in the large N limit, because of the central limit theorem, summing over the internal index β averages the system over an ensemble of configurations, and

$$\lim_{N\to\infty}\frac{1}{N}\sum_{\beta=1}^{N}\Phi_\beta^2(\mathbf{x},t) = \langle\Phi_\beta^2(\mathbf{x},t)\rangle \equiv S(t)\,, \qquad (7)$$

where $\langle\cdots\rangle$ denotes the thermal averages.

Hence, dropping the label α and Fourier transforming to reciprocal space, the equation of motion for the order parameter reads

$$\frac{\partial\Phi(\mathbf{k},t)}{\partial t} = -\Gamma k^p[k^2 + Q(t)]\Phi(\mathbf{k},t) + \eta(\mathbf{k},t)\,, \qquad (8)$$

where $p = 0$ for NCOP while $p = 2$ for COP, and we have defined

$$Q(t) = r + gS(t) + \lambda S^2(t) \qquad (9)$$

with

$$S(t) = \int_{|k|<\Lambda}\frac{d^dk}{(2\pi)^d}C(\mathbf{k},t) \qquad (10)$$

and Λ is a momentum cutoff.

$C(\mathbf{k},t)$ is the equal time structure factor, central quantity in the study of the domain growth, given by

$$C(\mathbf{k},t) = \langle\Phi(\mathbf{k},t)\Phi(-\mathbf{k},t)\rangle \qquad (11)$$

since we assume $\langle\Phi(\mathbf{k},0)\rangle = 0$ and the model is not symmetry breaking. $Q(t)$ is to be determined self-consistently from Eqs. (9), (10), and the equation of motion for $C(\mathbf{k},t)$:

$$\frac{\partial C(\mathbf{k},t)}{\partial t} = -2\Gamma k^p[k^2 + Q(t)]C(\mathbf{k},t) + 2\Gamma k^p T_f\,. \qquad (12)$$

In the following, the dynamical process of a zero-temperature ($T_f = 0$) quench will be studied, because thermal fluctuations are expected to play a minor role for deep quenches into the ordered phase. We shall also consider processes with long range correlated initial conditions, of the generic form

$$C(\mathbf{k},0) = \frac{\Delta}{k^\theta}\,. \tag{13}$$

$\theta = 0$ describes a quench from very high temperature that is a completely disordered state, whereas $\theta = 2$, a quench from a critical state. We stress that in order to avoid infrared divergences, d must always be greater than θ.

3. Solution of NCOP the Model

We now turn to the explicit solution of the equation of motion for the structure factor, Eq. (12), in the nonconserved case.

Upon introducing the characteristic length $L(t) = (2\Gamma t)^{\frac{1}{2}}$ and the scaling function $F(\mathbf{x}) = e^{-x^2}/x^\theta$ of the dimensionless wave vector $\mathbf{x} = \mathbf{k}\,L(t)$, Eq. (12) can be formally integrated, yielding

$$C(\mathbf{k},t) = \Delta\, e^{-2\Gamma Q(t)} L^\theta(t) F(\mathbf{x})\,, \tag{14}$$

where the function $Q(t)$ is obtained self-consistently, eliminating $S(t)$ from Eq. (9) with the help of Eq. (10). By integrating Eq. (12) for small t we conclude that $S(t)$ is a function of time given by

$$\frac{\dot S(0)}{S(0)} = -2\Gamma\left[\Lambda^2 \frac{(d-\theta)}{(d-\theta+2)} + r + gS(0) + \lambda S^2(0)\right] \tag{15}$$

for $t \to 0$, whereas for greater times, from Eq. (10), we obtain

$$S(t) = a\,\Delta\, L(t)^{\theta-d}\, e^{-2\Gamma Q(t)}\,, \tag{16}$$

where a is a constant.

In order to solve the equation of motion for $C(k,t)$ we need some knowledge about the asymptotic behavior of $S(t)$. In some cases (e.g. μ_3 or μ_6 case of Fig. 1) it is not *a priori* evident whether, under particular conditions, the relaxation into the metastable $\Phi \equiv 0$ configuration is allowed. As stressed in the introduction, the possibility of a decay into a state which is not a global minimum of the free energy is particularly interesting in this contest. Therefore we give here a criterion which allows us to establish if and when this happens:

A necessary and sufficient condition in order to have relaxation in the $\Phi \equiv 0$ final state (i.e. $\lim_{t\to\infty} S(t) = 0$) is to have

$$\left.\frac{\partial^2 V(\phi)}{\partial \phi^2}\right|_{\phi=0} \geq 0 \tag{17}$$

and

$$S(\bar t) \leq \frac{-g - \sqrt{g^2 - 4\lambda r}}{2\lambda} \tag{18}$$

at a time instant $\bar t$ such that the solution (16) is already valid.

We demonstrate the criterion beginning with the necessary condition (17), i.e

$$\left\{\lim_{t\to\infty} S(t) = 0\right\} \Rightarrow \left\{\left.\frac{\partial^2 V(\phi)}{\partial \phi^2}\right|_{\phi=0} \geq 0\right\}. \tag{19}$$

In fact, since

$$\left.\frac{\partial^2 V(\phi)}{\partial \phi^2}\right|_{\phi=0} = r\,, \tag{20}$$

we consider two cases.

(i) $r \neq 0$:

In this case, assuming $\lim_{t\to\infty} S(t) = 0$, Eq. (9) reads

$$\dot Q(t) \simeq r \tag{21}$$

for long times. Therefore asymptotically one has

$$S(t) \sim t^{\frac{\theta-d}{2}} e^{-2\Gamma r t} \tag{22}$$

and hence it must be $r > 0$ in order to have consistency.

(ii) $r = 0$:

In this case Eq. (9) becomes asymptotically

$$\dot Q(t) \simeq gS(t) \sim t^{\frac{\theta-d}{2}} e^{-2\Gamma Q(t)}\,. \tag{23}$$

Eliminating $Q(t)$, using Eq. (16), one obtains

$$S(t) \sim \frac{t^{\frac{\theta-d}{2}}}{b + c t^{\frac{\theta-d+2}{2}}}\,, \tag{24}$$

a result consistent with $\lim_{t\to\infty} S(t) = 0$ for every θ and d.

Secondly, for what concerns sufficiency, both Eqs. (17) and (18) are required to be simultaneously fulfilled. In fact, from Eqs. (18) and (9), we observe that $\dot Q(\bar t) \geq 0$. Furthermore from Eqs. (9) and (16) we obtain

$$2\Gamma \dot Q(\bar t) = \frac{\theta - d}{2\bar t} - \frac{\dot S(\bar t)}{S(\bar t)}\,. \tag{25}$$

Therefore $\dot Q(\bar t) \geq 0$ implies $\dot S(\bar t) < 0$, since $S(t)$ is positively defined. This is true also for every $t \geq \bar t$, hence $\lim_{t\to\infty} S(t) = 0$ (q.e.d).

As seen from Eqs. (17) and (18) the possibility of metastable relaxation is essentially due to a local property of the functional form of the potential around the metastable solution, i.e. Eq. (17), and to a dynamical condition, which is expressed by Eq. (18). In practice this condition can always be fulfilled, when Eq. (17) holds, by choosing an initial condition with $S(0)$ sufficiently small.

We now turn to consider different cases, according to the parameters $\mu \equiv (r, g, \lambda)$ characterizing $\mathcal{H}(\Phi)$.

Let us begin with the case of simple diffusion:

$$\mu_0 \equiv (\mathbf{r} = 0, \mathbf{g} = 0, \lambda = 0).$$

This is rather trivial but interesting because the presence of a fixed point at μ_0 often affects the behavior of the dynamical process characterized by different choices of the parameters μ, as is known from the Φ^4 theory.[4] Moreover the same scaling behavior is displayed in a case of metastable relaxation (i.e. μ_3).

Since $Q(t)$ is identically zero in this case, the structure factor results:

$$C(\mathbf{k}, t) = \Delta L^{\theta}(t) F(\mathbf{x}), \tag{26}$$

a form which is exact at all times (and for all N). In other words, scaling holds true from the beginning to the end, but differently from the standard scaling behavior observed in the late stages of phase separation (Eq. (1)), a memory of the initial condition always conserved in the scaling exponent θ in Eq. (26).

Next we consider the tricritical case:[5]

$$\mu_1 \equiv (\mathbf{r} = 0, \mathbf{g} = 0, \lambda > 0).$$

One obtains

$$C(\mathbf{k}, t) = \frac{\Delta L^{\theta}(t)}{\left[b + c\dfrac{\Delta^2 \lambda}{L(t)^{2(d-\tilde{d}_c)}}\right]^{\frac{1}{2}}} F(\mathbf{x}), \tag{27}$$

where b and c are integration constants, and $\tilde{d}_c = \theta + 1$ plays the role of a critical dimensionality. In fact, when $d > \tilde{d}_c$ the structure factor displays asymptotically the same behavior as in a quench at μ_0, with corrections to scaling. On the other hand, when $d < \tilde{d}_c$ there is a crossover: in the early stage, i.e. when the typical domain size $L(t)$ is less than $(\lambda\Delta^2)^{\frac{1}{2(d-\tilde{d}_c)}}$, the system still behaves as if it were a quench at μ_0, whereas for later times, asymptotically, the structure factor has the scaling form

$$C(\mathbf{k}, t) \sim L(t)^{d-1} F(\mathbf{x}). \tag{28}$$

Exactly at $d = \tilde{d}_c$, $C(\mathbf{k}, t)$ displays logarithmic corrections because of marginality:

$$C(\mathbf{k}, t) \sim \frac{L^{\theta}(t)}{\ldots} F(\mathbf{x}) \tag{29}$$

$$\mu_2 \equiv (\mathbf{r} = 0, \mathbf{g} > 0, \lambda > 0)$$

In this case, we expect the parameter λ to be irrelevant and the asymptotic behavior of the $C(\mathbf{k}, t)$ to be the same as for a Φ^4 theory (with $r = 0$). Therefore we write down briefly the solution (refer also to Ref. 4 for a discussion of this case). Since $\lim_{t\to\infty} S(t) = 0$, solving Eq. (9) for long times,

$$\dot{Q}(t) \simeq gS(t), \tag{30}$$

we find

$$C(\mathbf{k}, t) = \frac{\Delta L^{\theta}(t)}{b + c\dfrac{\Delta g}{L(t)^{d-d_c}}} F(\mathbf{x}), \tag{31}$$

where b and c are constants (different from the μ_1 case), and $d_c = \theta + 2$ is a critical dimensionality, which plays a role analogous to $\tilde{d}_c$ for the quench at μ_1.

$$\mu_3 \equiv (\mathbf{r} = 0, \mathbf{g} < 0, \lambda > 0)$$

This is a very peculiar case since the sufficient condition for the criterion, Eq. (17), suggests that metastable relaxation to $\Phi(\mathbf{r}, t = \infty) \equiv 0$ is not excluded, but Eq. (18) does not give us further information because it reduces to $S(\bar{t}) \le 0$ in this case, which is never true for finite times.

As a matter of fact, both the dynamics leading to stable and metastable equilibrium will be shown to be consistent with the model equations.

When stable equilibrium is reached, by letting $S(t) \to$ const asymptotically in Eq. (9), it is found that

$$C(\mathbf{k}, t) \simeq \frac{|g|}{\lambda} a \Delta L^{d}(t) F(\mathbf{x}) \tag{32}$$

and therefore standard scaling holds.

However, if the metastable solution is approached, by letting $S(t) \to 0$, Eq. (9) can be solved asymptotically as

$$\dot{Q}(t) \simeq -|g| S(t), \tag{33}$$

and therefore

$$e^{2\Gamma Q(t)} = -4\Gamma |g| a \frac{\Delta}{(d_c - d)} L(t)^{d_c - d} + c. \tag{34}$$

From Eq. (34) it is readily seen that the metastable solution is asymptotically consistent only for $d > d_c$. In this case we find

$$C(\mathbf{k}, t) = \frac{\Delta L^{\theta}(t)}{\ldots} F(\mathbf{x}). \tag{35}$$

The dynamics leading to metastability is the same as at $\mu = \mu_2$ and asymptotically the structure factor scales as in a quench at the trivial fixed point of μ_0 (with corrections to scaling). In this sense both g and λ are irrelevant, when $d > d_c$. However, even for $d \leq d_c$ the system follows the same dynamical path (35) for short times, since the singularities of Eq. (34) show up only for $L(t) \geq (\Delta|g|)^{-\frac{1}{d_c - d}}$. For $|g|$ or Δ small this regime can be rather long.

In summary, for $d \leq d_c$ the stable state is always reached by means of Eq. (32), after a transient, while, when $d > d_c$, both stable and metastable solutions are possible and the choice depends on Δ. If Δ is sufficiently small, stable solutions cannot be seen and the system relaxes to $\Phi \equiv 0$. If Δ is large, however, the fluctuations present in the initial conditions drive the system to stable equilibrium. Analogously if θ is large ($\theta \geq \theta_c = d - 2$) huge fluctuations of the order parameter at small momenta surely lead to stable equilibrium, which is not true for small θ ($\theta < \theta_c$).

$$\mu_4 \equiv (\mathbf{r} < 0, \mathbf{g}, \lambda > 0)$$

In this case, solving Eq. (9) by letting $S(t) \to \text{const}$ results in

$$C(\mathbf{k}, t) \sim \Delta L^d(t) F(\mathbf{x}) . \tag{36}$$

Standard scaling holds. No metastability occurs since the necessary condition (17) is not satisfied.

$$\mu_5 \equiv \left(\mathbf{r} > 0, \mathbf{g} \geq -4\sqrt{\frac{\lambda \mathbf{r}}{3}}, \lambda > 0 \right)$$

In the late stage

$$\dot{Q}(t) \simeq r \tag{37}$$

and

$$C(\mathbf{k}, t) \sim \Delta e^{-rL^2(t)} L^\theta(t) F(\mathbf{x}) . \tag{38}$$

Scaling is suppressed by the exponential damping of the structure factor.

$$\mu_6 \equiv \left(\mathbf{r} > 0, \mathbf{g} < -4\sqrt{\frac{\lambda \mathbf{r}}{3}}, \lambda > 0 \right)$$

According to statements (17) and (18) if the initial condition is such that $S(0)$ is sufficiently small, metastable relaxation in $\Phi \equiv 0$ is surely expected. In this case the equations can be solved asymptotically as in a quench at μ_5. The same result is obtained:

$$C(\mathbf{k}, t) \sim \Delta e^{-rL^2(t)} L^\theta(t) F(\mathbf{x}) . \tag{39}$$

We observe that the stronger character of metastability, as compared to the μ_3 case, eliminates the critical dimensionality d_c. Metastable relaxation happens in any dimension, for $S(0)$ small. Since Eq. (18) is only a sufficient condition, Eq. (39) can in principle hold asymptotically even for $S(t)$ larger than $(|g| - \sqrt{g^2 - 4\lambda r})/2\lambda$.

When $S(0)$ is sufficiently large, however, stable equilibrium must be obtained. In this case, letting $S(t) \to \text{const}$, we obtain

$$C(\mathbf{k}, t) \sim \Delta L^d(t) F(\mathbf{x}) \tag{40}$$

in the late stage. Standard scaling is thus recovered.

4. Summary

In this paper we have discussed the dynamical behavior of the spherical Φ^6 model in the nonconserved case, at zero temperature. As compared to the closely related Φ^4 model, the same asymptotic behaviors are found whenever the two models approach stable equilibrium and the potential has the same number of minima. Apart from the sector $r > 0$ of the phase diagram (see Fig. 1), where an exponential relaxation

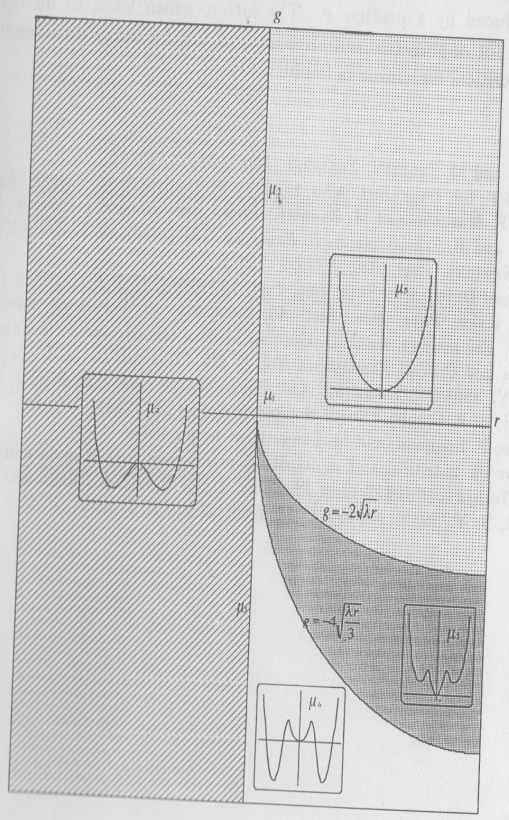

Fig. 1. Various shapes of the potential $V(|\phi|)$ are schematically shown as a function of the parameters $\mu \equiv [r, g, \lambda]$, for $\lambda > 0$.

towards the state $\Phi \equiv 0$ can be observed, the structure factor always obeys the standard scaling form (1), with different exponents. The fixed point at μ_0, which is the trivial one with $\alpha = \theta$, and the one of the ordering region, with $\alpha = d$, are particularly relevant to our discussion. The former can be attractive on the whole $r = 0$ axis, above a critical dimension d_c. Below d_c this sector behaves, for $g > 0$, as a transition zone towards the ordering region. This feature brings about new scaling exponents in the late stage while scaling controlled by μ_0 still holds for short times. On the other hand, the negative part of the $r = 0$ axis belongs to the region of the ordered states. Therefore the trivial fixed point (which represents now metastability) competes with the fixed point of the ordering region. The former prevails asymptotically for $d > d_c$ but controls the dynamics for short times even at $d < d_c$. Finally, the μ_6 region behaves as a transition sector between μ_4 and μ_5. The standard scaling of the ordered region is found together with the exponential relaxation induced by a positive r. The latter, which leads to metastability, is achieved only for appropriate initial conditions but for every dimensionality and, therefore, no critical dimension is found.

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