Comment on "Entropy Production and Fluctuation Theorems for Active Matter"

In Ref. [1], Mandal, Klymko, and DeWeese challenge the existing formula for entropy production for a renowned model (AOUP in their Letter) of active particles [2,3]. In this Comment, we question the central results of Ref. [1].

First, we show that the main result in [1], that is, their Eq. (9), is not correct. In order to prove this, one may directly analyze Eq. (2a) of [1], without the need of any transformation of variables. This circumvents the indeterminacy of the time-reversal operation for the Gaussian random force, $\mathbf{v}_i$, which is equal to the sum of two quantities having opposite time-reversal parities, as Eq. (2a) declares. The path probabilities induced by the colored noise are calculated without any ambiguity; see [4,5]. Such a calculation, in the case of a single particle in one dimension, $\Gamma(t) \equiv x(t)$, returns

$$P[\Gamma] \propto \exp\left(-\frac{1}{2}\int dt \int ds v(t) T^{-1}(t-s) v(s)\right), \quad (1)$$

where we have defined $T^{-1}(t) = (1/2D)\delta(t)[1-\tau^2(d^2/dt^2)]$ in such a way that $\int ds' T^{-1}(t-s') \times \langle v(s')v(s)\rangle = \delta(t-s)$. With algebra, one gets the following formula for the entropy production $\Sigma[\Gamma] = \log(P[\Gamma]/P[\Gamma^r])$, i.e., the only possible prescription for the AOUP system [and, in fact, Eq. (7) of [1] has not an equivalent in the overdamped equation]:

$$\Sigma[\Gamma] = -\mu \int_0^t ds \dot{x}(s)(T^{-1} * \Phi')(s) + \Phi'(s)(T^{-1} * \dot{x})(s), \quad (2)$$

where $*$ stands for the convolution operation. Performing algebraic manipulations, one finally gets

$$\Sigma[\Gamma] = \text{b.t.} + \frac{\mu\tau^2}{2D}\int^t \dot{x}^3(s)\Phi'''(s) ds, \quad (3)$$

where b.t. denotes boundary terms. This result is clearly different from Eq. (9) of [1]. Neglecting the b.t., Eq. (3) coincides with the results in Refs. [2,3], which have been obtained through the underdamped mapping [analogous to Eqs. (3) of [1]] and adopting a different time-reversal operation for the nonequilibrium force. Remarkably, Eq. (3) does not require any prescription of such kind: For this reason, it seems to us indisputable.

Second, we contest the identification of $-p/(\mu m) + \sqrt{2/(\mu\beta)}\eta$ with a thermal bath, which follows from a crucial confusion between $p = m\dot{x}$ and real particles' momentum. Based on this, the authors of [1] state that the total energy is $E = p^2/2m + \Phi(x)$. However, the AOUP system is different: It is described by an overdamped equation where the real particles' mass and momentum are unknown and their kinetic energy, in general, is not $p^2/2m$. The heat, the choice of the time reversal, and the whole "derivation" of Eq. (7) [6] all stem from such a wrong identification of mass, momenta, and kinetic energy. A consequence is seen when the potential Φ is removed, e.g., by considering a single particle and no external forces. In this case, the average heat exchange, Eq. (5) of [1], and entropy production, Eq. (9) of [1], both vanish even when $\tau > 0$: The model results at equilibrium even if it describes an active particle.

Third, we observe that a central application of their main result, the detailed fluctuation relation, Eq. (12) in [1], cannot be verified in experiments, since it involves a measurement of $P^r(\Sigma^r)$, the probability of entropy production according to a different dynamics, Eq. (7), which is not known to represent any realizable system.

After having shown how to remove certain ambiguities in the AOUP model, we sketch their origin [3,7]. The AOUP model represents only a coarse-grained level of description, where the variables $\{\dot{x}_i\}$ are not the real velocities of the particles and the fluctuating force (white noise) of the *real* thermal bath has been neglected. For this reason, there is no way to take into account the energetic and entropic exchanges with the physical thermostat, and it is not surprising to observe zero entropy production in some special cases, even if physically one expects it to be positive: That is a possible outcome of coarse-graining [5,8,9], which occurs also in [1] when $\Phi = 0$. This is, however, not a reason to identify the nonconservative self-propulsion force, or part of it, as a thermal bath force, a choice which is physically wrong and leads—as we showed—to inconsistent results. A derivation of an entropy production formula without resorting to any arbitrary prescription for time reversal, as sketched in Eq. (3) above, is the simplest way to settle the dispute.

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