Microscopic theory for negative differential mobility in crowded environments

Alessandro Sarracino

in collaboration with
O. Bénichou, P. Illien, G. Oshanin, R. Voituriez

Outline of the talk

▶ Introduction on negative differential mobility (NDM)

▶ The model: driven tracer in a lattice gas (ASEP in a sea of SEPs)
  - Physical argument for NDM at low density

▶ General expression for the force-velocity relation
  - Analytical solution and the decoupling approximation
  - Criterion for NDM in the parameter space

▶ Transition rates out of equilibrium
  - NDM and fluctuation-dissipation relations

▶ Conclusions and perspectives
Passive and active microrheology

Rheological properties in soft matter from the microscopic motion of colloidal tracers

Puertas & Voigtmann (2014), Squires & Mason (2010)

**Passive:** probes freely diffusing in the host medium due to thermal fluctuations

Stokes-Einstein relation \( D = \frac{k_B T}{6\pi \eta a} \)

Extension to the case where the probe size is comparable to the interaction length scales relevant for the host

**Active:** tracer particle (TP) driven by an external force \( F \) (pulling with a constant force, or dragging at constant velocity)

Linear response connects active and passive microrheology

Extensions to the non-linear response regime \( F \gg \frac{k_B T}{a} \approx \text{pN} \)
Passive and active microrheology

Rheological properties in soft matter from the microscopic motion of colloidal tracers

Puertas & Voigtmann (2014), Squires & Mason (2010)

**Passive:** probes freely diffusing in the host medium due to thermal fluctuations

Stokes-Einstein relation \( D = \frac{k_B T}{6\pi \eta a} \)

Extension to the case where the probe size is comparable to the interaction length scales relevant for the host

**Active:** tracer particle (TP) driven by an external force \( F \) (pulling with a constant force, or dragging at constant velocity)

Linear response connects active and passive microrheology

Extensions to the non-linear response regime \( F \gg \frac{k_B T}{a} \approx \text{pN} \)

Applications: complex fluids, gels, glasses, living cells, granular systems,…

Experimental techniques: optical and magnetic tweezers, Janus particles, etc…
Tracer particle (TP) driven by an external force $F$ in a host medium.

The differential mobility $\mu(F) = \frac{\delta V}{\delta F}\bigg|_F$ measures how the velocity increases with changing $F \rightarrow F + dF$. 

**Negative differential mobility**
Negative differential mobility

Tracer particle (TP) driven by an external force $F$ in a host medium

The differential mobility $\mu(F) = \frac{\delta V}{\delta F} \bigg|_F$ measures how the velocity increases with changing $F \rightarrow F + dF$

At equilibrium, Einstein relation $\mu(F = 0) = \beta D(F = 0)$
Negative differential mobility

Tracer particle (TP) driven by an external force $F$ in a host medium

The differential mobility $\mu(F) = \frac{\delta V}{\delta F} \bigg|_{F}$ measures how the velocity increases with changing $F \rightarrow F + dF$

At equilibrium, Einstein relation $\mu(F = 0) = \beta D(F = 0)$

Nonlinear response regime: increasing the applied force can reduce the probe’s drift velocity in the force direction $\mu(F) \leq 0$

“Getting more from pushing less” (Zia et al. Am. J. Phys. 2002)
Driven tracer in a hard-core lattice gas

General many-particle interacting system, analytically tractable

(N-1) hard-core particles, symmetric exclusion process, average waiting time $\tau^*$

Tracer driven by a force $F$, asymmetric exclusion process, average waiting time $\tau$

Particle density $\rho = \frac{N}{V}$
Driven tracer in a hard-core lattice gas

General many-particle interacting system, analytically tractable

(N-1) hard-core particles, symmetric exclusion process, average waiting time $\mathcal{T}$

Tracer driven by a force $\mathbf{F}$, asymmetric exclusion process, average waiting time $\mathcal{T}^*$

Particle density $\rho = \frac{N}{V}$

Tracer jump probabilities

$$p_\nu = \frac{e^{(\beta/2)\mathbf{F} \cdot \mathbf{e}_\nu}}{\sum_\mu e^{(\beta/2)\mathbf{F} \cdot \mathbf{e}_\mu}}$$

Local detailed balance

$$\frac{p_1}{p_{-1}} = e^{\beta \mathbf{F}}$$

LDB does not determine univocally the transition rates

Tracer driven by a force $\mathbf{F}$
Study of the **force-velocity relation** $V(F)$ and NDM phenomenon

Previous results in specific cases:

**Fixed obstacles** (Lorentz gas)

\[ \tau^*/\tau = \infty \]

analytic results at **low density**

Leitmann & Franosch PRL 2013

**Mobile obstacles**

\[ \tau^*/\tau < \infty, \; \rho = 0.2 \]

numerical analysis

Study of the force-velocity relation $V(F)$ and NDM phenomenon

Previous results in specific cases:

**Fixed obstacles (Lorentz gas)**

$\tau^*/\tau = \infty$

analytic results at low density
Leitmann & Franosch PRL 2013

**Mobile obstacles**

$\tau^*/\tau < \infty$, $\rho = 0.2$

numerical analysis

General description in all regimes?

Role of density and time scales ratio? Physical mechanism?
Argument for NDM at low density

Strong external force \( \epsilon = 2e^{-\beta F/2} \ll 1 \)

\( p_1 = 1 - \epsilon \) \quad \( p_{-1} = O(\epsilon^2) \) \quad \( p_{\mu \neq \pm 1} = \frac{\epsilon}{2d - 2} \)

Force-velocity relation:

\[ V(F) = \frac{\text{mean distance}}{\text{mean time of free flight} + \text{mean trapping time}} \]

Bénichou et al. PRL 2014
Argument for NDM at low density

Strong external force \( \epsilon = 2e^{-\beta F/2} \ll 1 \)

\[
p_1 = 1 - \epsilon \\
p_{-1} = O(\epsilon^2) \\
p_{\mu \neq \pm 1} = \frac{\epsilon}{2d - 2}
\]

Force-velocity relation:

\[
V(F) = \frac{\text{mean distance}}{\text{mean time of free flight} + \text{mean trapping time}}
\]

Mean distance between two obstacles \( 1/\rho \)

Mean duration of free flight \( \tau/[\rho(1 - \epsilon)] \)

Bénichou et al. PRL 2014
Argument for NDM at low density

Strong external force \( \epsilon = 2e^{-\beta F/2} \ll 1 \)

\[
p_1 = 1 - \epsilon \quad p_{-1} = O(\epsilon^2) \quad p_{\mu \neq \pm 1} = \frac{\epsilon}{2d - 2}
\]

Force-velocity relation:

\[
V(F) = \frac{\text{mean distance}}{\text{mean time of free flight} + \text{mean trapping time}}
\]

Mean distance between two obstacles \( 1/\rho \)

Mean duration of free flight \( \tau/[\rho(1 - \epsilon)] \)

\[
1/\tau_{\text{trap}} = \frac{3}{4\tau^*} + \frac{\epsilon}{\tau}
\]

obstacle steps tracer steps in a transverse direction

away
Argument for NDM at low density

\[ V(F) = \frac{1 - \epsilon}{\tau + 4\rho(1 - \epsilon) \frac{\tau^*}{3 + 4\epsilon\tau^* / \tau}} \]

Criterion for NDM \( \tau^* \gtrsim \tau / \sqrt{\rho} \)
Arguing for NDM at low density

\[ V(F) = \frac{1 - \epsilon}{\tau + 4\rho(1 - \epsilon)\frac{\tau^*}{3 + 4\epsilon \tau^*}} \]

Criterion for NDM \( \tau^* \gtrsim \tau / \sqrt{\rho} \)

Physical mechanism: a large force
- reduces the flight time between two consecutive encounters with bath particles;
- increases the escape time from traps created by surrounding obstacles.
\[ V(F) = \frac{1 - \epsilon}{\tau + 4\rho(1 - \epsilon) \frac{\tau^*}{3 + 4\epsilon\tau^*/\tau}} \]

**Criterion for NDM** \[ \tau^* \gtrsim \tau / \sqrt{\rho} \]

**Physical mechanism:** a large force reduces the flight time between two consecutive encounters with bath particles; increases the escape time from traps created by surrounding obstacles.

For \( \tau^* \) large enough ("slow" obstacles), traps are sufficiently long lived to slow down the TP when \( F \) is increased — NDM.
Master equation of the driven lattice gas

**Master Equation** for $P(R_{TP}, \eta; t)$

$R_{TP}$  tracer position

$\eta$  obstacle configuration

$$\partial_t P(R_{TP}, \eta; t) = \frac{1}{2d\tau^*} \sum_{\mu=1}^{d} \sum_{r \neq R_{TP} - e_\mu, R_{TP}} \left[ P(R_{TP}, \eta^r, \mu; t) - P(R_{TP}, \eta; t) \right]$$

$$+ \frac{1}{\tau} \sum_{\mu=1}^{d} p_\mu \{ [1 - \eta(R_{TP})] P(R_{TP} - e_\mu, \eta; t) \}$$

$$- \{ [1 - \eta(R_{TP} + e_\mu)] P(R_{TP}, \eta; t) \}$$
Master equation of the driven lattice gas

**Master Equation** for \( P(R_{TP}, \eta; t) \)

\[ \partial_t P(R_{TP}, \eta; t) = \frac{1}{2d\tau^*} \sum_{\mu=1}^{d} \sum_{r \neq R_{TP} - e_{\mu}, R_{TP}} \left[ P(R_{TP}, \eta^{r,\mu}; t) - P(R_{TP}, \eta; t) \right] \]

\[ + \frac{1}{\tau} \sum_{\mu=1}^{d} p_{\mu} \{ [1 - \eta(R_{TP})] P(R_{TP} - e_{\mu}, \eta; t) \} \]

\[ - [1 - \eta(R_{TP} + e_{\mu})] P(R_{TP}, \eta; t) \}

**Tracer velocity** \( V(F) \equiv \frac{d\langle R_{TP} \cdot e_1 \rangle}{dt} = \frac{1}{2d\tau^*} (A_1 - A_{-1}) \)

\[ A_{\nu} \equiv 1 + \frac{2d\tau^*}{\tau} p_{\nu} (1 - k(e_{\nu})) \)
Decoupling approximation and analytic solution

Density profile around the tracer

\[ k(\lambda; t) = \sum_{R_{TP}, \eta} \eta(R_{TP} + \lambda) P(R_{TP}, \eta; t) \]

occupation variable

Equation of motion for the density profile

\[ 2d\tau^* \partial_t k(\lambda; t) = \sum_{\mu} \left( \nabla_{\mu} - \delta_{\lambda, e_{\mu}} \nabla_{-\mu} \right) k(\lambda; t) \]

\[ + \frac{2d\tau^*}{\tau} \sum_{\nu} p_{\nu} \langle [1 - \eta(R_{TP} + e_{\nu})] \nabla_{\nu} \eta(R_{TP} + \lambda) \rangle \]

higher order correlations are involved
Decoupling approximation and analytic solution

Density profile
\[ k(\lambda; t) = \sum_{R_{TP}, \eta} \eta(R_{TP} + \lambda) P(R_{TP}, \eta; t) \]
around the tracer

occupation variable

Equation of motion for the density profile
\[
2d\tau^* \partial_t k(\lambda; t) = \sum_{\mu} \left( \nabla_\mu - \delta_\lambda, e_\mu \nabla_{-\mu} \right) k(\lambda; t)
\]
\[
+ \frac{2d\tau^*}{\tau} \sum_\nu p_\nu \langle [1 - \eta(R_{TP} + e_\nu)] \nabla_\nu \eta(R_{TP} + \lambda) \rangle
\]
higher order correlations are involved

Decoupling approximation
\[
\langle \eta(R_{TP} + \lambda) \eta(R_{TP} + e_\nu) \rangle \approx \langle \eta(R_{TP} + \lambda) \rangle \langle \eta(R_{TP} + e_\nu) \rangle
\]
for \( \lambda \neq e_\nu \)
Tracer velocity \( V(F) \equiv \frac{d\langle R_{TP} \cdot e_1 \rangle}{dt} = \frac{1}{2d\tau^*} (A_1 - A_{-1}) \)

The decoupling approximation allows us to obtain a closed nonlinear system of equations

\[
A_\nu = 1 + \frac{2d\tau^*}{\tau} p_\nu \left[ 1 - \rho - \rho (A_1 - A_{-1}) \frac{\det C_\nu}{\det C} \right]
\]

\[
C \equiv (A_\mu \nabla_{-\mu} \mathcal{F}_{e_\nu} - \alpha \delta_{\mu,\nu})_{\mu,\nu} \quad \quad \alpha = \sum_\mu A_\mu
\]

\[
C_\nu = C \rightarrow ( (\nabla_1 - \nabla_{-1}) \mathcal{F}_{e_\nu} )_\nu
\]

\[
\mathcal{F}_n = \left( \frac{A_{-1}}{A_1} \right)^{n_1/2} \int_0^\infty e^{-t} \text{I}_{n_1} (2\alpha^{-1} \sqrt{A_1 A_{-1}} t) \prod_{i=2}^d \text{I}_{n_i} (2\alpha^{-1} A_2 t) dt
\]
Decoupling approximation and analytic solution

Tracer velocity \( V(F) \equiv \frac{d\langle R_{TP} \cdot e_1 \rangle}{dt} = \frac{1}{2d\tau^*} (A_1 - A_{-1}) \)

The decoupling approximation allows us to obtain a closed nonlinear system of equations

\[
A_\nu = 1 + \frac{2d\tau^*}{\tau} \rho_\nu \left[ 1 - \rho - \rho(A_1 - A_{-1}) \frac{\det C_\nu}{\det C} \right]
\]

\[
C \equiv (A_\mu \nabla_{-\mu} F_{e_\nu} - \alpha \delta_{\mu,\nu})_{\mu,\nu} \quad \alpha = \sum_{\mu} A_\mu
\]

\[
C_\nu = C \rightarrow ((\nabla_1 - \nabla_{-1}) F_{e_\nu})_\nu
\]

\[
F_n = \left( \frac{A_{-1}}{A_1} \right)^{n_1/2} \int_0^\infty e^{-t} I_{n_1} (2\alpha^{-1} \sqrt{A_1 A_{-1}} t) \prod_{i=2}^d I_{n_i} (2\alpha^{-1} A_2 t) dt
\]

Solution for \( V(F) \) for arbitrary values of the parameters

Bénichou et al. PRL 2014
Linearized solution at low density

Low density limit $\rho \to 0$  Auxiliary variable $k(e_\nu) = \rho(1 + v_{e_\nu})$

$$V = \frac{1}{\tau}(p_1 - p_{-1}) - \frac{\rho}{\tau}(p_1 - p_{-1} + p_1 v_1 - p_{-1} v_{-1})$$
Linearized solution at low density

Low density limit \( \rho \rightarrow 0 \)  
Auxiliary variable \( k(e_{\nu}) = \rho(1 + v_{e_{\nu}}) \)

\[
V = \frac{1}{\tau}(p_1 - p_{-1}) - \frac{\rho}{\tau}(p_1 - p_{-1} + p_1 v_1 - p_{-1} v_{-1})
\]

Linear system of equations

\[
2d(1 + \frac{\tau^*}{\tau})v_n \quad = \quad \sum_{\nu = \pm 1, 2} [1 + 2d\frac{\tau^*}{\tau}p_{\nu}]v_{e_{\nu}} \nabla_{-\nu}F_n
\]

\[
- \quad 2d\frac{\tau^*}{\tau}(p_1 - p_{-1})(\nabla_1 - \nabla_{-1})F_n
\]
Linearized solution at low density

Low density limit $\rho \rightarrow 0$ Auxilary variable $k(e_\nu) = \rho(1 + v_{e_\nu})$

$$V = \frac{1}{\tau}(p_1 - p_{-1}) - \frac{\rho}{\tau}(p_1 - p_{-1} + p_1 v_1 - p_{-1} v_{-1})$$

Linear system of equations

$$2d(1 + \frac{\tau^*}{\tau})v_n = \sum_{\nu = \pm 1, 2} \left[ 1 + 2d\frac{\tau^*}{\tau}p_{\nu} \right] v_{e_\nu} \nabla_{-\nu} F_n$$

$$- 2d\frac{\tau^*}{\tau}(p_1 - p_{-1})(\nabla_1 - \nabla_{-1})F_n$$

For $\tau^* = \infty$ we recover the solution of the Lorentz lattice gas

Leitmann & Franosch PRL 2013
Linearized solution at low density

**Low density limit** \( \rho \to 0 \)  
**Auxiliary variable** \( k(e_\nu) = \rho(1 + v_{e_\nu}) \)

\[
V = \frac{1}{\tau} (p_1 - p_{-1}) - \frac{\rho}{\tau} (p_1 - p_{-1} + p_1 v_1 - p_{-1} v_{-1})
\]

**Linear system of equations**

\[
2d(1 + \frac{\tau^*}{\tau})v_n = \sum_{\nu=\pm 1,2} [1 + 2d\frac{\tau^*}{\tau}p_\nu]v_{e_\nu} \nabla_{-\nu} F_n
\]

\[
- 2d\frac{\tau^*}{\tau} (p_1 - p_{-1})(\nabla_1 - \nabla_{-1})F_n
\]

For \( \tau^* = \infty \) we recover the solution of the Lorentz lattice gas

Leitmann & Franosch PRL 2013

Exact at low density, for arbitrary \( \tau^* \)
Explicit **criterion** for NDM in the parameter space
Linearized solution at low density

Explicit criterion for NDM in the parameter space

Strong force $p_1 = 1 - \epsilon \quad p_{-1} = O(\epsilon^2) \quad p_{\mu \neq \pm 1} = \frac{\epsilon}{2d - 2}$

$$V \left( \frac{\tau^*}{\tau} \right) = V^{(0)} \left( \frac{\tau^*}{\tau} \right) + \epsilon V^{(1)} \left( \frac{\tau^*}{\tau} \right)$$

The sign of $V^{(1)} \left( \frac{\tau^*}{\tau} \right)$ determines the region of NDM
Linearized solution at low density

Explicit criterion for NDM in the parameter space

Strong force \[ p_1 = 1 - \epsilon \quad p_{-1} = O(\epsilon^2) \quad p_{\mu \neq \pm 1} = \frac{\epsilon}{2d - 2} \]

\[
V \left( \frac{\tau^*}{\tau} \right) = V^{(0)} \left( \frac{\tau^*}{\tau} \right) + \epsilon V^{(1)} \left( \frac{\tau^*}{\tau} \right)
\]

The sign of \( V^{(1)} \left( \frac{\tau^*}{\tau} \right) \) determines the region of NDM

Exact asymptotic result

\[
\rho \xrightarrow{\frac{\tau^*}{\tau} \to \infty} \frac{1}{4 \left( \frac{\tau^*}{\tau} \right)^2}
\]
Explicit solution at high density

High density limit $\rho \rightarrow 1$

$$A_\nu = 1 + \frac{4\tau^*}{\tau} p_\nu [1 - \rho(2 + k_\nu)]$$
Explicit solution at high density

High density limit $\rho \rightarrow 1$

$$A_\nu = 1 + \frac{4\tau^*}{\tau} p_\nu [1 - \rho(2 + k_\nu)]$$

Tracer velocity

$$V(\rho \rightarrow 1) = \frac{1}{\tau} (p_1 - p_{-1})(1 - \rho) \frac{1}{1 + \frac{4\tau^*}{\tau} \frac{(p_1 + p_{-1})(4 - 8/\pi)}{8/\pi}}$$

Exact result

$$V(F) = \frac{1}{\tau} (1 - \rho) \frac{\sinh(\beta F/2)}{1 + \cosh(\beta F/2)[1 + \frac{2\tau^*}{\tau}(\pi - 2)]}$$
Explicit solution at high density

High density limit $\rho \to 1$ \quad $A_\nu = 1 + \frac{4\tau^*}{\tau} p_\nu [1 - \rho(2 + k_\nu)]$

Tracer velocity

$$V(\rho \to 1) = \frac{1}{\tau} (p_1 - p_{-1})(1 - \rho) \frac{1}{1 + \frac{4\tau^*}{\tau} \frac{(p_1 + p_{-1})(4 - 8/\pi)}{8/\pi}}$$

Exact result \quad $V(F) = \frac{1}{\tau} (1 - \rho) \frac{\sinh(\beta F/2)}{1 + \cosh(\beta F/2)[1 + \frac{2\tau^*}{\tau}(\pi - 2)]}$

For equal time scales \quad $\tau^* = \tau$

Bénichou & Oshanin PRE (2002)
Explicit solution at high density

High density limit $\rho \rightarrow 1$

$$A_\nu = 1 + \frac{4\tau^*}{\tau} p_\nu [1 - \rho (2 + k_\nu)]$$

Tracer velocity

$$V(\rho \rightarrow 1) = \frac{1}{\tau} (p_1 - p_{-1}) (1 - \rho) \frac{1}{1 + \frac{4\tau^*}{\tau} \frac{(p_1 + p_{-1})(4 - 8/\pi)}{8/\pi}}$$

Exact result

$$V(F) = \frac{1}{\tau} (1 - \rho) \frac{\sinh(\beta F/2)}{1 + \cosh(\beta F/2) [1 + \frac{2\tau^*}{\tau} (\pi - 2)]}$$

For equal time scales $\tau^* = \tau$

Bénichou & Oshanin PRE (2002)
Comparison with Monte Carlo numerical simulations

d = 2, \tau = 1

(a) \rho = 0.05
(b) \tau^* = 10
(c) \rho = 0.999
(d) \rho = 0.5

Very good agreement in a wide range of parameters.
Criterion for negative differential mobility

The analytical solution allows us to obtain a complete description.

Phase chart in the parameter space:

time scales $\frac{\tau^*}{\tau}$ and density $\rho$

Physical mechanism: coupling between density and time scales ratio
Transition rates out of equilibrium

Decoupling approximation  →  General solution

Tracer velocity  \( V(F) \equiv \frac{d\langle \mathbf{R}_{TP} \cdot \mathbf{e}_1 \rangle}{dt} = \frac{1}{2d\tau^*}(A_1 - A_{-1}) \)

\[ A_\nu = 1 + \frac{2d\tau^*}{\tau} p_\nu \left[ 1 - \rho - \rho(A_1 - A_{-1}) \frac{\text{det} C_\nu}{\text{det} C} \right] \]

Significant dependence on the choice of transition probabilities?
Transition rates out of equilibrium

General form of transition rates

\[ k(x, y) = \psi(x, y)e^{S(x,y)/2}\delta(K.C.) \]

- \[ \psi(x, y) = \psi(y, x) \geq 0 \]  \hspace{1cm} Symmetric (kinetic) part

- \[ S(x, y) = -S(y, x) \]  \hspace{1cm} Antisymmetric part
Transition rates out of equilibrium

General form of transition rates

\[ k(x, y) = \psi(x, y) e^{S(x, y)/2} \delta (K.C.) \]

\[ \psi(x, y) = \psi(y, x) \geq 0 \quad \text{Symmetric (kinetic) part} \]

\[ S(x, y) = -S(y, x) \quad \text{Antisymmetric part} \]

Local detailed balance imposes a constraint on the antisymmetric part

\[ S(x, y) \propto \text{entropy flux} \quad \rightarrow \quad S(x, x + e_\nu) = \beta F \cdot e_\nu \]
Transition rates out of equilibrium

General form of transition rates

\[ k(x, y) = \psi(x, y) e^{S(x, y)/2} \delta(K.C.) \]

- \( \psi(x, y) = \psi(y, x) \geq 0 \)  \text{Symmetric (kinetic) part}
- \( S(x, y) = -S(y, x) \)  \text{Antisymmetric part}

Local detailed balance imposes a constraint on the antisymmetric part

\[ S(x, y) \propto \text{entropy flux} \rightarrow S(x, x + e_\nu) = \beta F \cdot e_\nu \]

Arbitrary choice for the symmetric part

- Leitmann & Franosch, Bénichou et al.
- Basu & Maes

\[ \psi(x, x + e_\nu) = \begin{cases} 
1/\tau [e^{\beta F/2} + e^{-\beta F/2} + 2] & \text{for } \nu = \pm 1 \\
1/4\tau & \text{for } \nu = \pm 2 
\end{cases} \]

independent of \( F \) in the transverse direction
Role of the transition probabilities

\[ p_\nu = \frac{e^{(\beta/2) \mathbf{F} \cdot e_\nu}}{\sum_\mu e^{(\beta/2) \mathbf{F} \cdot e_\mu}} \]

(Leitmann & Franoch, Bénichou et al.)

\[ p_\uparrow = p_\downarrow = \frac{1}{4} \text{ independent of } \mathbf{F} \]

(Basu & Maes)

One obstacle can create a long lived trap

No trapping effect at linear order in the density
Role of the transition probabilities

\[ p_\nu = \frac{e^{(\beta/2)\mathbf{F} \cdot \mathbf{e}_\nu}}{\sum_\mu e^{(\beta/2)\mathbf{F} \cdot \mathbf{e}_\mu}} \]

(Leitmann & Franoch, Bénichou et al.)

\[ p^\uparrow = p^\downarrow = \frac{1}{4} \quad \text{independent of F} \]

(Basu & Maes)

One obstacle can create a long lived trap

No trapping effect at linear order in the density

Different choices \rightarrow significant macroscopic differences

Problem: how to define microscopic transition rates out of equilibrium? (e.g. molecular motors with external load)
Fluctuation-Dissipation Relation

Linear response around nonequilibrium

Trajectory \( \omega \equiv \{x_s\}_{s=0}^{s=t} \) characterized by discrete jumps at \( s_i \)
and by exponentially distributed waiting times \( s_{i+1} - s_i \)

Entropy flux
\[
\Sigma(\omega) = \sum_i S(x_{s_i}, x_{s_{i+1}})
\]

Dynamical activity ("frenesy")
\[
D(\omega) = \int_0^t ds \left( \sum_y k(x_s, y) \right) - \sum_i \log \psi(x_{s_i}, x_{s_{i+1}})
\]

Nonequilibrium FDR
\[
\frac{d\langle O \rangle_F}{dF} = \frac{1}{2} \left\langle O \frac{d\Sigma}{dF} \right\rangle_F - \left\langle O \frac{dD}{dF} \right\rangle_F
\]
(Baiesi, Maes, Wynants PRL 2009)
Fluctuation-Dissipation Relation

In our case:

\[
\Sigma(\omega) = \beta F (N_{\rightarrow} - N_{\leftarrow})
\]

\[
D(\omega) = \int_0^t ds \left\{ p_1 [1 - \eta(x_s + e_1)] + p_{-1} [1 - \eta(x_s + e_{-1})] + p_2 [1 - \eta(x_s + e_2)] + p_{-2} [1 - \eta(x_s + e_{-2})] \right\} - N \log \left[ 1 / (e^{\beta F / 2} + e^{-\beta F / 2} + 2) \right]
\]

jumps on the right \quad \text{and} \quad \text{jumps on the left}
Fluctuation-Dissipation Relation

In our case:

\[
\Sigma(\omega) = \beta F (N_\rightarrow - N_\leftarrow)
\]

\[
D(\omega) = \int_0^t ds \left\{ p_1 [1 - \eta (x_s + e_1)] + p_{-1} [1 - \eta (x_s + e_{-1})] \right\}
+ \left\{ p_2 [1 - \eta (x_s + e_2)] + p_{-2} [1 - \eta (x_s + e_{-2})] \right\}
- N \log [1 / (e^{\beta F/2} + e^{-\beta F/2} + 2)]
\]

Consider \( O \equiv V \) in

\[
\frac{d\langle O \rangle_F}{dF} = \frac{1}{2} \left\langle O \frac{d\Sigma}{dF} \right\rangle_F - \left\langle O \frac{dD}{dF} \right\rangle_F
\]

Differential mobility, linear response around nonequilibrium

\[
\frac{d\langle V \rangle_F}{dF} = \frac{\beta}{2} \langle V^2 \rangle_{F,c} - p'_1 \langle V \cdot (t - t_\rightarrow) \rangle_{F,c} - p'_{-1} \langle V \cdot (t - t_\leftarrow) \rangle_{F,c}
- 2p'_2 \langle V \cdot (t - t_{\uparrow}) \rangle_{F,c} - h' \langle V \cdot N \rangle_{F,c}
\]
Conclusions

- **Microscopic theory** for NDM in a driven lattice gas model:
  - Decoupling approximation
  - **General expression** for the force-velocity relation
  - **Exact** at low and high density
  - Unification of recent results

- **Criterion** for NDM in the parameter space:
  - **Coupling** between density and diffusion time scales

- **Role** of transition rates out of equilibrium
  - **Significant** macroscopic effects
Perspectives

» Analytical expression of velocity fluctuations and higher order moments

- How to infer the applied force from a velocity measurement?

» Nonequilibrium fluctuation-dissipation relations

- Linear FDR around nonequilibrium
- Analytical expressions for the terms responsible for NDM
Perspectives

» Analytical expression of velocity fluctuations and higher order moments
  How to infer the applied force from a velocity measurement?

» Nonequilibrium fluctuation-dissipation relations
  Linear FDR around nonequilibrium
  Analytical expressions for the terms responsible for NDM

» Is it possible to observe NDM in off-lattice systems?
  Recent studies show a monotonic behavior
  To explore a wider range of parameters (for tracer and obstacles)

» Experiments and simulations in driven granular systems?

» Role of the kinetic part of transition rates out of equilibrium
  To measure “effective” transition rates from molecular dynamics
Negative differential mobility in different systems

- **Nonequilibrium steady states**

- **Models of Brownian motors**
  (Cecchi & Magnasco PRL 1996, $\langle v \rangle$

- **Kinetically constraint models for glassy dynamics**
  (Jack et al. PRE 2008, Sellitto PRL 2008)