Exercises for the course Statistical Physics

Note

The student must be able to solve exercises of the level of the following ones

Es. 1 Derive the following relations

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A{\cup}B{\cup}C) = P(A){+}P(B){+}P(C){-}P(A{\cap}B){-}P(A{\cap}C){-}P(B{\cap}C){+}P(A{\cap}B{\cap}C)$

Es. 2 Given two independent variables X and Y taking values $k = 1, 2, ...$ with probability 2^{-k} compute the probability that

a) $Y = X$

b) $X = 3Y$

Es. 3 The PdF $p_x(x)$ of the variable x is zero, but in the interval $(0, 1)$ where it is constant, compute the PdF of $y = 4x(1 - x)$.

Es. 4 Given the independent variables $x_1, x_2, ..., x_N$ distributed according the same PdF $p_x(x)$, compute the PdF of the variable

$$y_N = max\{x_1, x_2, ..., x_N\} \, .$$

Es. 5 Let x_1 and x_2 independent gaussian variables with zero average and unitary variance,

• compute the Pdf of the variables

$$r = \sqrt{x_1^2 + x_2^2} \ , \ z = \frac{x_2}{x_1}$$

• show that the variables

$$y_1 = x_1 + x_2 \ , \ y_2 = x_1 - x_2$$

are independent and gaussian.

Es. 6 Let $x_1, ..., x_N$ independent gaussian variables with zero average and unitary variance,
- compute the Pdf of the variables

$$y_N = \sum_{n=1}^{N} x_n^2 \ , \ z_N = \sqrt{y_N}$$

for $N = 2$ and $N = 3$.

Es. 7 The PdF of the variables x and y is

$$p_{xy}(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} exp - \frac{x^2 + y^2 - 2\rho xy}{2(1-\rho^2)}$$

where $|\rho| < 1$, compute the marginal PdF $p_x(x) = \int p_{xy}(x, y)dy$ and the conditional PdF $p(y|x) = p_{xy}(x, y)/p_x(x)$ at varying ρ.

Es 8 X and Y are independent poissonian variables with parameter λ_1 and λ_2 respectively, i.e.

$$P(X = k) = \frac{\lambda_1^k}{k!} e^{-\lambda_1} \ , \ P(Y = k) = \frac{\lambda_2^k}{k!} e^{-\lambda_2} \ , \ k \geq 0 \,,$$

compute
a) $P(X + Y = k)$ with $k \geq 0$;
b) $P(X = k|X + Y = n)$ with $k \geq 0$ and $n \geq k \geq 0$.

Es. 9 Let x_1 and x_2 independent gaussian variables with mean value m_1 and m_2 and variance σ_1^2 and σ_2^2, compute the PdF of $y = ax_1 + bx_2 + c$ where a, b and c are fixed real numbers.

Es. 10 Given the independent variables $x_1, x_2, ..., x_N$ distributed according the same PdF $p_x(x)$,

$$p_x(x) = \frac{1}{\pi(1 + x^2)}$$

compute the PdF of the variable

$$y_N = \frac{1}{N} \sum_{n=1}^{N} x_n \ .$$

Discuss the result in the limit $N \gg 1$.
Suggestion Remind that

$$\int \frac{e^{itx}}{\pi(1+x^2)} dx = e^{-|t|} \ .$$

Es. 11 Let us consider the following Langevin equation

$$\frac{dx}{dt} = a(x) + \sqrt{2c}\,\eta$$

where η is a standard white noise i.e. $< \eta(t) >= 0$ and $< \eta(t)\eta(t') >= \delta(t-t')$,
• show that the hypothesis for the validity of the Fokker- Planck equation are valid:

$$\lim_{\Delta t \to 0} \frac{< \Delta x(\Delta t)|x >}{\Delta t} = a(x) \ ,$$

$$\lim_{\Delta t \to 0} \frac{< (\Delta x(\Delta t))^2|x >}{\Delta t} = 2c \ ,$$

$$\lim_{\Delta t \to 0} \frac{< (\Delta x(\Delta t))^n|x >}{\Delta t} = 0 \ , \ se \ n \geq 3\, ,$$

• write the corresponding Fokker- Planck;
• discuss the conditions on $a(x)$ in order to have an invariant probability density of the Fokker- Planck equation.

Es. 12 Given the following Langevin equation

$$\frac{dv}{dt} = -\frac{v}{\tau} + \sqrt{2c}\,\eta$$

where η is a standard white noise i.e. $< \eta(t) >= 0$ and $< \eta(t)\eta(t') >= \delta(t-t')$,
• write the Fokker-Planck equation;
• determine c in such a way $< v^2 >= k_B T/m$;
• compute $< v(t)|v(0) >$ and $< v(t)^2|v(0) >$;
• compute $< v(t)v(0) >$;

• give the $p(v,0)$ find $p(v,t)$, discuss the limit $t \gg \tau$;
• compute $< x(t)|x(0) >$ and $< x(t)^2|x(0) >$, where $x(t) = x(0) + \int_0^t v(t')dt'$;
• using for τ the Stokes law for shere of mass m and radius a in a fluid with temperature T and viscosity $\tilde{\eta}$, derive the Einstein relation.

Es. 13 Consider the following discrete time stochastic process

$$v_{n+1} = av_n + w_n$$

where $|a| < 1$ and the variables $\{w_n\}$ are independent, with zero average and a know PdF $p(w)$:
• given a generic $p(v,0)$ show that, if $p(w)$ is gaussian, for $n \to \infty$ $p(v,n)$ approaches to a gaussian.
• given a generic $p(v,0)$ show that $n \to \infty$, $p(v,n)$ approaches to $p_*(v)$, if $p(w)$ is not a gaussian, $p_*(v)$ must be different from a gaussian;
• compute $< v_n v_0 >$.
Suggestion Use characteristic functions and cumulants.

Es. 14 Consider the following discrete time stochastic process

$$x_{n+1} = x_n + v_n$$

where the variables $\{v_n\}$ have zero average and are NOT independent, and the correlation function $c_n =< v_n v_0 >$ is known
• in the case $\sum_0^\infty c_n < \infty$ find the diffusion coefficient

$$D = \lim_{n\to\infty} \frac{< (x_n - x_0)^2 >}{2n} .$$

Es. 15 The position of a particle evolves with the following rule
• at the time 0, $x(0) = 0$, it is extracted a time τ_1 with a PdF $P(\tau)$ which goes to zero faster than τ^{-3}; up to τ_1 one has

$$x(t) = v_1 t$$

where $v_1 = \pm v$ with probability 1/2
• at the time τ_1, it is extracted a new time τ_2 with the same PdF $P(\tau)$, for $\tau_1 < t < \tau_1 + \tau_2$ on has

$$x(t) = x(\tau_1) + v_2(t - \tau_1)$$

where $v_2 = \pm v$ with probability 1/2.
And so on: for $T_n < t < T_n + \tau_{n+1}$ one has

$$x(t) = x(T_n) + v_{n+1}(t - T_n) \quad where\ T_n = \sum_{j=1}^{n} \tau_j$$

$v_n = \pm v$ with probability 1/2.
Show that

$$D = \lim_{t \to \infty} \frac{< x(t)^2 >}{2t} = \frac{v^2}{2} \frac{< \tau^2 >}{< \tau >} \ .$$

Suggestion Consider the time T_n with $n \gg 1$ and remind the law of the large numbers.

Es. 16 Given the deterministic map $x_{t+1} = 2x_t \ mod\, 1$,
• find the invariant probability density;
• at $t = 0$ the PdF is zero but in $[x^*, x^* + \Delta]$ where is constant; study numerically for different values of x^* and Δ the time evolution of $p(x, t)$, verify that for $t \to \infty$ $p(x, t)$ approaches to $p_{inv}(x)$.

Es. 17 Given the deterministic map on $[0, 1]$: $x_{t+1} = 4x_t(1 - x_t)$,
• verify that the invariant PdF is

$$p_{inv}(x) = \frac{1}{\pi\sqrt{x(1-x)}}$$

• at $t = 0$ the PdF is zero but in $[x^*, x^* + \Delta]$ where is constant; study numerically for different values of x^* and Δ the time evolution of $p(x, t)$, verify that for $t \to \infty$, $p(x, t)$ approaches to $p_{inv}(x)$.

Es.18 Given a Markov chain with the detailed balance, show the validity of the following relations:
• $P_{i\to j}P_{j\to k}P_{k\to i} = P_{i\to k}P_{k\to j}P_{j\to i}$
• $P_{i\to j}P_{j\to k}P_{k\to m}P_{m\to i} = P_{i\to m}P_{m\to k}P_{k\to j}P_{j\to i}$
• $< g(t)f(0) >=< g(0)f(t) >$

where g is g_i if the system is in the state i, in the same way f is f_j if the system is in the state j, and

$$< g(t)f(0) >= \sum_{i,j} f_i g_j P_i P_{i\to j}(t)$$

being $\{P_i\}$ the invariant probabilities and $P_{i\to j}(t)$ the probability to have a transition from i to j in t steps.

Es. 19 Given a Markov chain with two states:

$$P_{1\to 1} = 1 - p \ , P_{1\to 2} = p \ , P_{2\to 1} = q \ , P_{2\to 2} = 1 - q,$$

where $0 < q < 1$ and $0 < p < 1$,
• find the invariant probabilities;
• show that the detailed balance holds;
• compute $< g(t)g(0) >$, where g is 1 if $i = 1$ and 0 if $i = 2$.

Es. 20 Given the master equation with two states

$$\frac{dp_1}{dt} = -ap_1 + bp_2 \ , \ \frac{dp_2}{dt} = ap_1 - bp_2 \ ,$$

with $a > 0$ and $b > 0$:
• given $p_1(0)$, compute $p_1(t)$;
• verify the validity of the detailed balance at any time. i.e:

$$p_1^{inv} p_{1\to 2}(t) = p_2^{inv} p_{2\to 1}(t) \, ,$$

• compute the correlation function $\langle f(t)f(0)\rangle$ where $f = 1$ if the system is in the state 1 and $f = -1$ if the system is in the state 2.
The correlation function is defined as

$$\langle f(t)f(0)\rangle = \sum_i f_i f_j p_i^{inv} p_{i\to j}(t) \, .$$