5 Laplace's Method

Laplace's method is useful when trying to estimate integrals of the form

$$I(\lambda) = \int_a^b e^{-\lambda p(t)} q(t)\, dt,$$

where a, b may be finite or infinite.

The following technique dates back to Laplace (1820). Observe that the peak value of the function $e^{-\lambda p(t)}$ occurs at the point $t = t_0$ where $p(t)$ is a minimum. For large λ the peak is concentrated in a neighbourhood of $t - t_0$, see for example Fig. 1 where a plot of the function $e^{-\lambda(\cosh(t)-1)}$ is shown for varying λ.

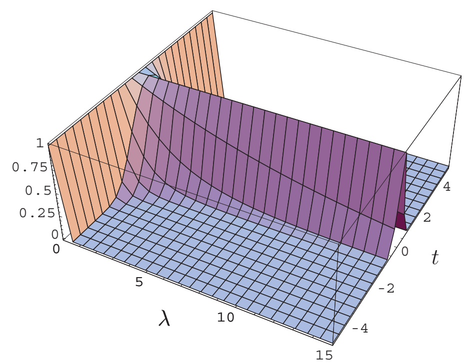

Figure 1: Plot of $f(\lambda, t) = e^{-\lambda \cosh[t]} e^{\lambda}$. Observe peak is concentrated near $t = 0$.

In essence Laplace's method is as follows: Suppose that $t_0 = a$ and $p'(a) > 0, q(a) \neq 0$. In the integral

$$I(\lambda) = \int_a^b e^{-\lambda p(t)} q(t)\, dt,$$

we replace $p(t), q(t)$ by local series expansions near $t = t_0$. Then

$$I(\lambda) \sim \int_a^b e^{-\lambda(p(a)+p'(a)(t-t_0))} q(a)\, dt.$$

We replace the upper-limit by ∞ to obtain

$$I(\lambda) \sim q(a) e^{-\lambda p(a)} \int_a^\infty e^{-\lambda(t-a)p'(a)}\, dt.$$

Hence

$$I(\lambda) \sim q(a) \frac{e^{-\lambda p(a)}}{\lambda p'(a)}.$$

If instead $t = t_0$ is an interior point and $p''(t_0) > 0$ then

$$I(\lambda) = \int_a^b e^{-\lambda p(t)} q(t)\, dt \sim \int_a^b e^{-\lambda(p(t_0)+\frac{1}{2}p''(t_0)(t-t_0)^2)} q(t_0)\, dt \qquad (5.1)$$

Since the peak is concentrated in the neighbourbood of $t - t_0$ we may replace the upper and lower limits in (5.1) by $\pm\infty$ with neglible error. Then using $\int_{-\infty}^{\infty} e^{-at^2}\, dt = \sqrt{\pi/a}$ for $a > 0$ we obtain,

$$I(\lambda) \sim e^{-\lambda p(t_0)} q(t_0) \int_{-\infty}^{\infty} e^{-\lambda \frac{(t-t_0)^2}{2} p''(t_0)}\, dt = e^{-\lambda p(t_0)} q(t_0) \sqrt{\frac{2\pi}{\lambda p''(t_0)}}.$$

These hand waving arguments work remarkably well and are proven more formally below.

Theorem *Suppose*

1. *$p(t) > p(a)$ for $t \in (a, b)$ and the minimum of $p(t)$ is only approached at $t = a$.*

2. *$p'(t), q'(t)$ are continuous in a neighbourhood of $t = a$ except possibly at $t = a$.*

3. *As $t \to a+$*

$$p(t) \sim p(a) + \sum_{k=0}^{\infty} p_k (t-a)^{k+\mu}, \quad q(t) \sim \sum_{k=0}^{\infty} q_k (t-a)^{k+\sigma-1},$$

where $\mu > 0, Re(\sigma) > 0, p_0 \neq 0, q_0 \neq 0$. Also we assume that we can differentiate $p(t)$ to obtain

$$p'(t) \sim \sum_{k=0}^{\infty} (k+\mu) p_k (t-a)^{k+\mu-1}.$$

4. *$\int_a^b e^{-\lambda p(t)} q(t)\, dt$ converges absolutely for large λ.*

Then

$$I(\lambda) = \int_a^b e^{-\lambda p(t)} q(t)\, dt \sim e^{-\lambda p(a)} \sum_{k=0}^{\infty} \Gamma\left(\frac{k+\sigma}{\mu}\right) \frac{a_k}{\lambda^{\frac{k+\sigma}{\mu}}},$$

where $v = p(t) - p(a)$ and

$$f(v) = \frac{q(t)}{p'(t)} \sim \sum_{k=0}^{\infty} a_k v^{\frac{k+\sigma-\mu}{\mu}} \quad as \quad v \to 0+.$$

Proof

Let $v = p(t) - p(a)$ then

$$I(\lambda) = \int_a^b e^{-\lambda p(t)} q(t)\, dt$$

$$= e^{-\lambda p(a)} \int_0^{p(b)-p(a)} e^{-\lambda v} f(v)\, dv$$

where $f(v) = q(t)/p'(t)$. Hence

$$I(\lambda) = e^{-\lambda p(a)} \int_0^{\infty} e^{-\lambda v} f(v)\, dv - e^{-\lambda p(a)} \int_{p(b)-p(a)}^{\infty} e^{-\lambda v} f(v)\, dv. \tag{5.2}$$

The contribution from the last integral in (5.2) can be shown to be negligble. If we use Watson's lemma for the other integral noting that as $t \to a+$, $v \to 0$ and

$$f(v) \sim \sum_{k=0}^{\infty} a_k v^{\frac{k+\sigma-\mu}{\mu}}.$$

This gives

$$I(\lambda) \sim e^{-\lambda p(a)} \int_0^{\infty} e^{-\lambda v} \sum_{k=0}^{\infty} a_k v^{\frac{k+\sigma}{\mu}-1}\, dv$$

$$= e^{-\lambda p(a)} \sum_{k=0}^{\infty} \int_0^{\infty} e^{-\lambda v} a_k v^{\frac{k+\sigma}{\mu}-1}\, dv, .$$

Hence

$$I(\lambda) \sim e^{-\lambda p(a)} \sum_{k=0}^{\infty} a_k \Gamma\left(\frac{k+\sigma}{\mu}\right) \frac{1}{\lambda^{\frac{k+\sigma}{\mu}}}.$$

Example

Consider the modified Bessel function of the second kind

$$K_\nu(\lambda) = \int_0^{\infty} e^{-\lambda \cosh t} \cosh(\nu t)\, dt$$

and we need the behaviour for large λ.

Here $p(t) = \cosh t$ has a minimum value of 1 at $t = 0$. Hence put

$$v = \cosh t - 1$$

For small t

$$v = \frac{t^2}{2!} + \frac{t^4}{4!} + \dots. \tag{5.3}$$

We can invert this to find t as a function of v for v small and the leading term is $t = (2v)^{\frac{1}{2}}$. This suggests that for small v we may write,

$$t = (2v)^{\frac{1}{2}} + c_1 v + c_2 v^{\frac{3}{2}} + \dots.$$

Thus substituting into (5.3) we find

$$v = \frac{t^2}{2!} + \frac{t^4}{4!} + \dots$$

$$= \frac{1}{2}[(2v)^{\frac{1}{2}} + c_1 v + c_2 v^{\frac{3}{2}} + \dots]^2 + \frac{1}{4!}[(2v)^2 + \dots] + \dots,$$

$$= v + v^{\frac{3}{2}} c_1 \sqrt{2} + v^2[\sqrt{2}c_2 + \frac{c_1^2}{2} + \frac{1}{6}] + \dots.$$

Comparing like powers of v on both sides implies that

$$c_1 = 0, \quad c_2 = -\frac{1}{6\sqrt{2}}.$$

Hence

$$t = (2v)^{\frac{1}{2}} - \frac{1}{6\sqrt{2}} v^{\frac{3}{2}} + \dots.$$

Hence

$$K_\nu(\lambda) = \int_0^\infty e^{-\lambda \cosh t} \cosh(\nu t)\, dt = e^{-\lambda} \int_0^\infty e^{-\lambda v} \frac{dt}{dv}[1 + \frac{\nu^2}{2} t^2 + \dots]\, dv$$

$$= e^{-\lambda} \int_0^\infty e^{-\lambda v}[\frac{1}{2}\sqrt{2} v^{-\frac{1}{2}} - \frac{1}{4\sqrt{2}} v^{\frac{1}{2}} + \dots][1 + \frac{\nu^2}{2}(2v) + \dots]\, dv,$$

$$= e^{-\lambda} \int_0^\infty e^{-\lambda v}[\frac{\sqrt{2}}{2} v^{-\frac{1}{2}} + v^{\frac{1}{2}}(\frac{\sqrt{2}}{2}\nu^2 - \frac{1}{4\sqrt{2}}) + \dots]\, dv.$$

This gives

$$K_\nu(\lambda) = e^{-\lambda}\sqrt{\frac{\pi}{2\lambda}}\left[1 + \frac{1}{2}(\nu^2 - \frac{1}{4})\frac{1}{\lambda} + \dots\right],$$

as $\lambda \to \infty$.

Example- Stirling's formula for large x. We will show how Laplace's method can be used to estimate the Gamma function $\Gamma(\lambda)$ for large values of the argument. Consider

$$\Gamma(\lambda+1) = \lambda\Gamma(\lambda) = \int_0^\infty e^{-y} y^\lambda \, dy. \tag{5.4}$$

Hence

$$\Gamma(\lambda) = \frac{1}{\lambda}\int_0^\infty e^{-y} y^\lambda \, dy.$$

Now

$$e^{-y} y^\lambda = e^{-y+\lambda \log y},$$

and the function $r(y) = -y + \lambda \log y$ has a minimum at $y = \lambda$. It is better to work with a fixed point rather than one depending on λ. So put $y = \lambda t$. Then substituting into (5.4) gives

$$\Gamma(\lambda) = \frac{1}{\lambda}\int_0^\infty e^{-\lambda t}\lambda^\lambda t^\lambda \lambda \, dt,$$

$$= \lambda^\lambda \int_0^\infty e^{-\lambda(t-\log t)} \, dt.$$

Consider

$$I(\lambda) = \int_0^\infty e^{-\lambda(T-\log T)} \, dT.$$

Now $P(T) = T - \log T$ has a minimum value of 1 at $T = 1$ for $T > 0$. If we are interesed in just the dominant term for $\Gamma(x)$ we can replace $P(T)$ by a local expansion in the vicinity of $T = 1$ and work with that. Below we show how more terms can generated. First we write

$$I(\lambda) = \int_0^1 e^{-\lambda P(T)} \, dT + \int_1^\infty e^{-\lambda P(T)} \, dT, \tag{5.5}$$

and estimate the two integrals separately.

Consider

$$I_1 = \int_0^1 e^{-\lambda P(T)} \, dT. \tag{5.6}$$

Put $t = 1 - T$ in (5.6) so that the minimum occurs at $t = 0$ and then

$$I_1 = \int_0^1 e^{-\lambda(1-t-\log(1-t))} \, dt. \tag{5.7}$$

Next let

$$v = 1 - t - \log(1-t) - 1 = -t - \log(1-t).$$

For small t we have

$$v = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4} + \dots.$$

This suggests that for small v

$$t = (2v)^{\frac{1}{2}} + c_1 v + c_2 v^{\frac{3}{2}} + \dots.$$

Hence

$$v = \frac{1}{2}[(2v)^{\frac{1}{2}} + c_1 v + c_2 v^{\frac{3}{2}} + \dots]^2 + \frac{1}{3}[(2v)^{\frac{1}{2}} + c_1 v + \dots]^3 + \frac{1}{4}[(2v)^2 + \dots] + \dots,$$

$$= \frac{1}{2}[2v + 2\sqrt{2v}c_1 v + 2\sqrt{2v}c_2 v^{\frac{3}{2}} + c_1^2 v^2 + \dots]$$

$$+\frac{1}{3}[(2v)^{\frac{3}{2}} + 3(2v)(c_1 v + c_2 v^{\frac{3}{2}}) + \dots] + v^2 + \dots,$$

$$= v + v^{\frac{3}{2}}[\sqrt{2}c_1 + \frac{2\sqrt{2}}{3}] + v^2[\sqrt{2}c_2 + \frac{c_1^2}{2} + 2c_1 + 1] + \dots.$$

Equating like powers of v on both sides gives $c_1 = -\frac{2}{3}$ and

$$\sqrt{2}c_2 = -(1 + 2c_1 + \frac{c_1^2}{2}) = -(1 - \frac{4}{3} + \frac{2}{9}) = \frac{1}{9}.$$

Thus $c_2 = \frac{\sqrt{2}}{18}$ and we have

$$t = (2v)^{\frac{1}{2}} - \frac{2}{3}v + \frac{\sqrt{2}}{18}v^{\frac{3}{2}} + \dots.$$

This gives

$$\frac{dt}{dv} = \frac{1}{\sqrt{2}}v^{-\frac{1}{2}} - \frac{2}{3} + \frac{1}{6\sqrt{2}}v^{\frac{1}{2}} + \dots$$

as $v \to 0+$. With the substitution $v = -t - \log(1-t)$ the integral (5.5) becomes

$$I_1 = e^{-\lambda} \int_0^\infty e^{-\lambda v} \frac{dt}{dv}\, dv.$$

Using Watson's lemma means replacing $\frac{dt}{dv}$ by the expansion for small v to get

$$I_1(\lambda) \sim e^{-\lambda} \int_0^\infty e^{-\lambda v} \left[\frac{1}{\sqrt{2}}v^{-\frac{1}{2}} - \frac{2}{3} + \frac{1}{6\sqrt{2}}v^{\frac{1}{2}} + \dots\right] dv,$$

$$= e^{-\lambda}\left[\sqrt{\frac{\pi}{2\lambda}} - \frac{2}{3\lambda} + \sqrt{\frac{\pi}{2}}\frac{1}{12\lambda^{\frac{3}{2}}} + \dots\right]. \quad (5.8)$$

We still need to consider the second of the integrals in (5.5), ie,

$$I_2 = \int_1^\infty e^{-\lambda(T - \log T)}\, dT = e^{-\lambda} \int_0^\infty e^{-\lambda(t - \log(1+t))}\, dt. \quad (5.9)$$

Here $p(t) = t - \log(1+t)$ has a minimum value of 0 at $t = 0$.Put $v = t - \log(1+t)$. As $t \to 0+$ we have

$$v = \frac{t^2}{2} - \frac{t^3}{3} + \frac{t^4}{4} + \ldots.$$

Inverting this for small v suggests that

$$t = (2v)^{\frac{1}{2}} + c_1 v + c_2 v^{\frac{3}{2}} + \ldots.$$

Thus

$$v = \frac{1}{2}[(2v)^{\frac{1}{2}} + c_1 v + c_2 v^{\frac{3}{2}} + \ldots]^2 - \frac{1}{3}[(2v)^{\frac{1}{2}} + c_1 v + \ldots]^3 + \frac{1}{4}[(2v)^2 + \ldots] + \ldots,$$

$$= \frac{1}{2}[2v + 2\sqrt{2v}c_1 v + 2\sqrt{2v}c_2 v^{\frac{3}{2}} + c_1^2 v^2 + \ldots]$$

$$-\frac{1}{3}[(2v)^{\frac{3}{2}} + 3(2v)(c_1 v + c_2 v^{\frac{3}{2}}) + \ldots] + v^2 + \ldots,$$

$$= v + v^{\frac{3}{2}}[\sqrt{2}c_1 - \frac{2\sqrt{2}}{3}] + v^2[\sqrt{2}c_2 + \frac{c_1^2}{2} - 2c_1 + 1] + \ldots.$$

Hence $c_1 = \frac{2}{3}$ and

$$\sqrt{2}c_2 = -(1 - 2c_1 + \frac{c_1^2}{2}) = -(1 - \frac{4}{3} + \frac{2}{9}) = \frac{1}{9}.$$

Thus $c_2 = \frac{\sqrt{2}}{18}$ and we have

$$t = (2v)^{\frac{1}{2}} + \frac{2}{3}v + \frac{\sqrt{2}}{18}v^{\frac{3}{2}} + \ldots.$$

This gives

$$\frac{dt}{dv} = \frac{1}{\sqrt{2}}v^{-\frac{1}{2}} + \frac{2}{3} + \frac{1}{6\sqrt{2}}v^{\frac{1}{2}} + \ldots$$

as $v \to 0+$. With the substitution $v = t - \log(1+t)$ the integral (5.9) for I_2 becomes

$$I_2 = e^{-\lambda}\int_0^\infty e^{-\lambda v}\frac{dt}{dv}\,dv.$$

$$I_2(\lambda) \sim e^{-\lambda}\int_0^\infty e^{-\lambda v}\left[\frac{1}{\sqrt{2}}v^{-\frac{1}{2}} + \frac{2}{3} + \frac{1}{6\sqrt{2}}v^{\frac{1}{2}} + \ldots\right]dv,$$

$$= e^{-\lambda}\left[\sqrt{\frac{\pi}{2\lambda}} + \frac{2}{3\lambda} + \sqrt{\frac{\pi}{2}}\frac{1}{12\lambda^{\frac{3}{2}}} + \ldots\right]. \quad (5.10)$$

Combining the two expressions (5.8),(5.10) for I_1 and I_2 shows that

$$\Gamma(\lambda) = \lambda^\lambda(I_1(\lambda) + I_2(\lambda)),$$

and using the derived asymptotic expansions for the two integrals gives

$$\Gamma(\lambda) \sim \lambda^{\lambda} e^{-\lambda} \sqrt{\frac{2\pi}{\lambda}} \left[1 + \frac{1}{12\lambda} + \dots\right],$$

as $\lambda \to \infty$.

This is Stirling's formula for the Gamma function for large values of the argument.

6 Method of stationary phase

In place of Laplace type integrals of the form (4.1) suppose we consider integrals of the form

$$I(\lambda) = \int_a^b e^{i\lambda p(t)} q(t)\, dt \tag{6.1}$$

and we require the behaviour of $I(\lambda)$ for large λ. A special case of these are Fourier transforms with a, b replaced by $\pm\infty$ and $p(t) = t$. For integrals of the form there is a famous result known as the **Riemann-Lebegue lemma** which states that $I(\lambda) \to 0$ as $\lambda \to \infty$ provided $|q(t)|$ is integrable in the interval $[a, b]$ and that $p(t)$ is continuously differentiable for $a \leq t \leq b$ and not constant on any subinterval in $a \leq t \leq b$.

If $p'(t)$ is non-zero in $a \leq t \leq b$ then we can use integration by parts and show that $I(\lambda) = O(1/\lambda)$ as $\lambda \to \infty$. The more interesting case is when $p'(t)$ is zero in $a \leq t \leq b$.

Observe that for large λ the integrand in (6.1) oscillates and contributions cancel out except near end points and near stationary points of $p(t)$. The behaviour of the integral can be estimated by looking at the local behaviour of the functions $p(t), q(t)$ near end points and near the stationary points of $p(t)$, as we did with Laplace's method. The basic idea of the method of stationary phase is as follows. Suppose that $p(t)$ has a single stationary point for at $t = t_0$ in $a < t < b$ and we can write

$$p(t) = p(t_0) + \frac{1}{2} p''(t_0)(t - t_0)^2 + \dots, \quad q(t) = q(t_0) + \dots.$$

Then we can approximate $I(\lambda)$ as

$$I(\lambda) \sim \int_{-\infty}^{\infty} e^{i\lambda(p(t_0) + \frac{1}{2}(t-t_0)^2 P''(t_0))} q(t_0)\, dt \sim e^{i\lambda p(t_0)} q(t_0) \int_{-\infty}^{\infty} e^{i\lambda \frac{p''(t_0)}{2} T^2}\, dT,$$

and so

$$I(\lambda) \sim \sqrt{\frac{2\pi}{\lambda}} e^{\frac{i\pi}{4}} e^{i\lambda p(t_0)} q(t_0),$$

where we have used

$$\int_{-\infty}^{\infty} e^{i\lambda T^2}\, dT = \sqrt{\frac{\pi}{\lambda}} e^{\frac{i\pi}{4}}.$$

The above can be generalised to deal with other behaviours and to obtain higher order behaviour as follows. Suppose that $p(t)$ has a single stationary point $t = t_0$ in $t \in [a, b]$. We can write

$$I(\lambda) = \int_a^{t_0} e^{i\lambda p(t)} q(t)\, dt + \int_{t_0}^{b} e^{i\lambda p(t)} q(t)\, dt. \tag{6.2}$$

Assume that near $t = t_0+$ we have

$$p(t) = p(t_0) + \alpha(t - t_0)^{\nu} + o((t - t_0)^{\nu}), \quad q(t) = \beta(t - t_0)^{\delta - 1} + o((t - t_0)^{\nu}), \tag{6.3}$$

where $\nu > 0, \delta > 0$, and that the expression for $p(t)$ is differentiable, ie

$$p'(t) \sim \alpha\nu(t - t_0)^{\nu - 1} \quad \text{as} \quad t \to t_0 + .$$

Consider

$$I_1(\lambda) = \int_{t_0}^{b} e^{i\lambda p(t)} q(t)\, dt.$$

If we make the substitution

$$v = s(p(t) - p(t_0)) \tag{6.4}$$

where $s = \text{sgn}(\alpha)$ then

$$I_1(\lambda) = e^{i\lambda p(t_0)} \int_0^{|p(b) - p(t_0)|} e^{is\lambda v} F(v)\, dv \tag{6.5}$$

where

$$F(v) = \frac{sq(t)}{p'(t)}.$$

Note that from (6.3), (6.4) as $t \to t_0+$

$$t - t_0 \sim \left(\frac{v}{|\alpha|}\right)^{\frac{1}{\nu}}.$$

Thus using the behaviour of $q(t)$ given in (6.3) we have

$$F(v) \sim \frac{s\beta(t - t_0)^{\delta - 1}}{\alpha\nu(t - t_0)^{\nu - 1}} \sim \frac{s\beta}{\alpha\nu} \left(\frac{v}{|\alpha|}\right)^{\frac{\delta}{\nu} - 1}.$$

If $F(v)$ is well behaved for large v then using the above we can approxmiate I_1 by

$$I_1(\lambda) = e^{i\lambda p(t_0)} \int_0^{|p(b) - p(t_0)|} e^{i\lambda sv} F(v)\, dv$$

$$\sim e^{i\lambda p(t_0)} \int_0^\infty e^{i\lambda s v} F(v)\, dv.$$

We can extract the leading order behavior of I_1 by replacing $F(v)$ with the local behaviour near $v \to 0+$. Thus

$$I_1(\lambda) \sim s e^{i\lambda p(t_0)} \int_0^\infty e^{i\lambda s v} \frac{\beta}{\alpha\nu} \left(\frac{v}{|\alpha|}\right)^{\frac{\delta}{\nu}-1} dv$$

$$\sim e^{i\lambda p(t_0)} \frac{s\beta}{\alpha\nu} e^{i\frac{\pi}{2}\frac{\delta}{\nu}s} \frac{\Gamma(\frac{\delta}{\nu})}{|\alpha|^{\frac{\delta}{\nu}-1}\lambda^{\frac{\delta}{\nu}}},$$

where we have used the result

$$\int_0^\infty e^{i\lambda\sigma t} t^{s-1}\, dt = \lambda^{-s} e^{i\sigma s\pi/2}\Gamma(s)$$

for $\lambda > 0$ and $\sigma = \pm 1$. Hence

$$I_1(\lambda) \sim e^{i\lambda p(t_0)} \frac{\beta}{\nu} e^{i\frac{\pi}{2}\frac{\delta}{\nu}s} \frac{\Gamma(\frac{\delta}{\nu})}{(|\alpha|\lambda)^{\frac{\delta}{\nu}}}. \tag{6.6}$$

Similarly for

$$I_2(\lambda) = \int_a^{t_0} e^{i\lambda p(t)} q(t)\, dt$$

suppose that as $t \to t_0-$

$$p(t) \sim p(t_0) + \gamma(t_0 - t)^\epsilon + o((t-t_0)^\epsilon), \qquad q(t) \sim \rho(t_0 - t)^{\sigma-1} + o((t-t_0)^\sigma),$$

where $\epsilon > 0, \sigma > 0$. Then

$$I_2(\lambda) \sim\sim e^{i\lambda p(t_0)} \frac{\rho}{\epsilon} e^{i\frac{\pi}{2}\frac{\sigma}{\epsilon}S} \frac{\Gamma(\frac{\sigma}{\epsilon})}{(|\gamma|\lambda)^{\frac{\sigma}{\epsilon}}}, \tag{6.7}$$

where $S = \mathrm{sgn}(\gamma)$.

The dominant contribution to I is given by adding the estimates (6.6), (6.7) for I_1 and I_2 to get

$$I(\lambda) \sim e^{i\lambda p(t_0)} \frac{\beta}{\nu} e^{i\frac{\pi}{2}\frac{\delta}{\nu}s} \frac{\Gamma(\frac{\delta}{\nu})}{(|\alpha|\lambda)^{\frac{\delta}{\nu}}} + e^{i\lambda p(t_0)} \frac{\rho}{\epsilon} e^{i\frac{\pi}{2}\frac{\sigma}{\epsilon}S} \frac{\Gamma(\frac{\sigma}{\epsilon})}{(|\gamma|\lambda)^{\frac{\sigma}{\epsilon}}}.$$

Near an end point one can adapt the above analysis as appropriate. The above ideas can be treated more formally, see, for example, chapter 6 of Olver.

Example Consider the Bessel function of order n where n is real

$$J_n(\lambda) = \frac{1}{\pi} \int_0^\pi \cos(nt - \lambda \sin t)\, dt.$$

We can write this as

$$J_n(\lambda) = \frac{1}{\pi}\Re\left[\int_0^{\pi} e^{int - i\lambda\sin t}\,dt\right].$$

Here $p(t) = \sin t$ has a single stationary point at $t = \frac{\pi}{2}$ for $t \in [0, \pi]$. First let $t = \frac{\pi}{2} + T$ and then

$$J_n(\lambda) = \int_{-\frac{\pi}{2}}^{0} + \int_0^{\frac{\pi}{2}} \left(e^{in(\frac{\pi}{2}+T)}e^{-i\lambda\cos T}\right)\,dT. \tag{6.8}$$

Consider

$$I_1 = \int_{-\frac{\pi}{2}}^{0} e^{in(\frac{\pi}{2}+T)}e^{-i\lambda\cos T}\,dT = e^{in\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} e^{-inT}e^{-i\lambda\cos T}\,dT.$$

Put

$$u = -\cos T + 1 \sim \frac{T^2}{2} + O(T^4) \quad \text{as} \quad T \to 0.$$

Inverting gives

$$T = (2u)^{\frac{1}{2}} + \dots \quad \text{as} \quad u \to 0+.$$

Thus

$$I_1 \sim e^{in\frac{\pi}{2}}\int_0^{\pi/2} e^{i\lambda(u-1)}(1+\dots)(2u)^{-\frac{1}{2}}\,du,$$

$$I_1 \sim e^{in\frac{\pi}{2}-i\lambda}\frac{1}{\sqrt{2}}\int_0^{\infty} e^{i\lambda u}u^{-\frac{1}{2}}\,du = e^{in\frac{\pi}{2}-i\lambda}e^{\frac{i\pi}{4}}\sqrt{\frac{\pi}{2\lambda}}. \tag{6.9}$$

Next consider

$$I_2 = \int_0^{\frac{\pi}{2}} e^{in(\frac{\pi}{2}+T)}e^{-i\lambda\cos T}\,dT.$$

Put

$$u = -\cos T + 1 \sim \frac{T^2}{2} \quad \text{as} \quad T \to 0+.$$

Thus

$$T = (2u)^{\frac{1}{2}} \quad \text{as} \quad u \to 0+.$$

Hence

$$I_2 \sim e^{in\frac{\pi}{2}}\int_0^{\pi/2}(1+\dots)e^{i\lambda(u-1)}(2u)^{-\frac{1}{2}}\,du,$$

$$\sim e^{in\frac{\pi}{2}-i\lambda}\int_0^{\infty} e^{i\lambda u}(2u)^{-\frac{1}{2}}\,du.$$

Thus

$$I_2 \sim e^{in\frac{\pi}{2}-i\lambda}e^{\frac{i\pi}{4}}\sqrt{\frac{\pi}{2\lambda}}. \tag{6.10}$$

Hence finally using (6.8), (6.9), (6.10) we obtain

$$J_n(\lambda) \sim \frac{1}{\pi}\Re\left[2e^{in\frac{\pi}{2}-i\lambda}e^{\frac{i\pi}{4}}\sqrt{\frac{\pi}{2\lambda}}+\dots\right]$$

$$=\sqrt{\frac{2}{\pi\lambda}}\cos(\frac{\pi}{4}+\frac{n\pi}{2}-\lambda) \quad \text{as} \quad \lambda\to\infty.$$